The Parma Polyhedra Library User's Manual* (version 0.6.1)

Roberto Bagnara†
Patricia M. Hill‡
Enea Zaffanella§

based on previous work also by

Elisa Ricci

and

Sara Bonini
Andrea Pescetti
Angela Stazzone
Tatiana Zolo

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†bagnara@cs.unipr.it, Department of Mathematics, University of Parma, Italy.
‡hill@comp.leeds.ac.uk, School of Computing, University of Leeds, U.K.
§zaffanella@cs.unipr.it, Department of Mathematics, University of Parma, Italy.

This document describes the Parma Polyhedra Library (PPL).

For the most up-to-date information see the Parma Polyhedra Library WWW site:

`http://www.cs.unipr.it/ppl/`

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1 General Information on the PPL

1.1 The Main Features

The Parma Polyhedra Library (PPL) is a modern C++ library for the manipulation of numerical information that can be represented by points in some n-dimensional vector space. For instance, one of the key domains the PPL supports is that of rational convex polyhedra (Section **Convex Polyhedra**(p. 2)). Such domains are employed in several systems for the analysis and verification of hardware and software components, with applications spanning imperative, functional and logic programming languages, synchronous languages and synchronization protocols, real-time and hybrid systems. Even though the PPL library is not meant to target a particular problem, the design of its interface has been largely influenced by the needs of the above class of applications. That is the reason why the library implements a few operators that are more or less specific to static analysis applications, while lacking some other operators that might be useful when working, e.g., in the field of computational geometry.

The main features of the library are the following:

- it is user friendly: you write `x + 2*y + 5*z <= 7` when you mean it;
- it is fully dynamic: available virtual memory is the only limitation to the dimension of anything;
- it provides full support for the manipulation of convex polyhedra that are not topologically closed;
- it is written in standard C++: meant to be portable;
- it is exception-safe: never leaks resources or leaves invalid object fragments around;
- it is rather efficient: and we hope to make it even more so;
- it is thoroughly documented: perhaps not literate programming but close enough;
- it has interfaces to other programming languages: including C and a number of Prolog systems;
- it is free software: distributed under the terms of the GNU General Public License.

In addition to the basic domains, we also provide generic support for constructing new domains from pre-existing domains. The following domains and domain constructors are provided by the PPL:

- the domain of topologically closed, rational convex polyhedra;
- the domain of rational convex polyhedra that are not necessarily closed;
- the powerset construction;
- the powerset construction, instantiated for rational convex polyhedra.

In the following sections we describe these domains and domain constructors together with their representations and operations that are available to the PPL user.

In the final section of this chapter (Section **Using the Library**(p. 13)), we provide some additional advice on the use of the library.

1.2 Convex Polyhedra

In this section we introduce convex polyhedra, as considered by the library, in more detail. For more information about the definitions and results stated here see **[BRZH02b]**(p. 15), **[Fuk98]**(p. 16), **[NW88]**(p. 17), and **[Wil93]**(p. 18).

Vectors, Matrices and Scalar Products

We denote by $\mathbb{R}^n$ the n-dimensional vector space on the field of real numbers $\mathbb{R}$, endowed with the standard topology. The set of all non-negative reals is denoted by $\mathbb{R}_+$. For each $i \in \{0, \ldots, n-1\}$, v_i denotes the i-th component of the (column) vector $\boldsymbol{v} = (v_0, \ldots, v_{n-1})^{\mathrm{T}} \in \mathbb{R}^n$. We denote by $\mathbf{0}$ the vector of $\mathbb{R}^n$, called *the origin*, having all components equal to zero. A vector $\boldsymbol{v} \in \mathbb{R}^n$ can be also interpreted as a matrix in $\mathbb{R}^{n\times 1}$ and manipulated accordingly using the usual definitions for addition, multiplication (both by a scalar and by another matrix), and transposition, denoted by $\boldsymbol{v}^{\mathrm{T}}$.

The *scalar product* of $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, denoted $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$, is the real number

$$\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = \sum_{i=0}^{n-1} v_i w_i.$$

For any $S_1, S_2 \subseteq \mathbb{R}^n$, the *Minkowski's sum* of S_1 and S_2 is: $S_1 + S_2 = \{\, \boldsymbol{v}_1 + \boldsymbol{v}_2 \mid \boldsymbol{v}_1 \in S_1, \boldsymbol{v}_2 \in S_2 \,\}$.

Affine Hyperplanes and Half-spaces

For each vector $\boldsymbol{a} \in \mathbb{R}^n$ and scalar $b \in \mathbb{R}$, where $\boldsymbol{a} \neq \boldsymbol{0}$, and for each relation symbol $\bowtie \in \{=, \geq, >\}$, the linear constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b$ defines:

- an affine hyperplane if it is an equality constraint, i.e., if $\bowtie \in \{=\}$;
- a topologically closed affine half-space if it is a non-strict inequality constraint, i.e., if $\bowtie \in \{\geq\}$;
- a topologically open affine half-space if it is a strict inequality constraint, i.e., if $\bowtie \in \{>\}$.

Note that each hyperplane $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = b$ can be defined as the intersection of the two closed affine half-spaces $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq b$ and $\langle -\boldsymbol{a}, \boldsymbol{x} \rangle \geq -b$. Also note that, when $\boldsymbol{a} = \boldsymbol{0}$, the constraint $\langle \boldsymbol{0}, \boldsymbol{x} \rangle \bowtie b$ is either a tautology (i.e., always true) or inconsistent (i.e., always false), so that it defines either the whole vector space $\mathbb{R}^n$ or the empty set $\varnothing$.

Convex Polyhedra

The set $\mathcal{P} \subseteq \mathbb{R}^n$ is a *not necessarily closed convex polyhedron* (*NNC polyhedron*, for short) if and only if either $\mathcal{P}$ can be expressed as the intersection of a finite number of (open or closed) affine half-spaces of $\mathbb{R}^n$ or $n = 0$ and $\mathcal{P} = \varnothing$. The set of all NNC polyhedra on the vector space $\mathbb{R}^n$ is denoted $\mathbb{P}_n$.

The set $\mathcal{P} \in \mathbb{P}_n$ is a *closed convex polyhedron* (*closed polyhedron*, for short) if and only if either $\mathcal{P}$ can be expressed as the intersection of a finite number of closed affine half-spaces of $\mathbb{R}^n$ or $n = 0$ and $\mathcal{P} = \varnothing$. The set of all closed polyhedra on the vector space $\mathbb{R}^n$ is denoted $\mathbb{CP}_n$.

When ordering NNC polyhedra by the set inclusion relation, the empty set $\varnothing$ and the vector space $\mathbb{R}^n$ are, respectively, the smallest and the biggest elements of both $\mathbb{P}_n$ and $\mathbb{CP}_n$. The vector space $\mathbb{R}^n$ is also called the *universe* polyhedron.

In theoretical terms, $\mathbb{P}_n$ is a *lattice* under set inclusion and $\mathbb{CP}_n$ is a *sub-lattice* of $\mathbb{P}_n$.

Bounded Polyhedra

An NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ is *bounded* if there exists a $\lambda \in \mathbb{R}_+$ such that

$$\mathcal{P} \subseteq \bigl\{\, \boldsymbol{x} \in \mathbb{R}^n \bigm| -\lambda \leq x_j \leq \lambda \text{ for } j = 0, \ldots, n-1 \,\bigr\}.$$

A bounded polyhedron is also called a *polytope*.

1.3 Representations of Convex Polyhedra

NNC polyhedra can be specified by using two possible representations, the constraints (or implicit) representation and the generators (or parametric) representation.

Constraints representation

In the sequel, we will simply write "equality" and "inequality" to mean "linear equality" and "linear inequality", respectively; also, we will refer to either an equality or an inequality as a *constraint*.

By definition, each polyhedron $\mathcal{P} \in \mathbb{P}_n$ is the set of solutions to a *constraint system*, i.e., a finite number of constraints. By using matrix notation, we have

$$\mathcal{P} \stackrel{\text{def}}{=} \{\, \boldsymbol{x} \in \mathbb{R}^n \mid A_1\boldsymbol{x} = \boldsymbol{b}_1, A_2\boldsymbol{x} \geq \boldsymbol{b}_2, A_3\boldsymbol{x} > \boldsymbol{b}_3 \,\},$$

where, for all $i \in \{1,2,3\}$, $A_i \in \mathbb{R}^{m_i} \times \mathbb{R}^n$ and $\boldsymbol{b}_i \in \mathbb{R}^{m_i}$, and $m_1, m_2, m_3 \in \mathbb{N}$ are the number of equalities, the number of non-strict inequalities, and the number of strict inequalities, respectively.

Combinations and Hulls

Let $S = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subseteq \mathbb{R}^n$ be a finite set of vectors. For all scalars $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, the vector $\boldsymbol{v} = \sum_{j=1}^{k} \lambda_j \boldsymbol{x}_j$ is said to be a *linear* combination of the vectors in S. Such a combination is said to be

- a *positive* (or *conic*) combination, if $\forall j \in \{1, \ldots, k\} : \lambda_j \in \mathbb{R}_+$;
- an *affine* combination, if $\sum_{j=1}^{k} \lambda_j = 1$;
- a *convex* combination, if it is both positive and affine.

We denote by $\operatorname{linear.hull}(S)$ (resp., $\operatorname{conic.hull}(S)$, $\operatorname{affine.hull}(S)$, $\operatorname{convex.hull}(S)$) the set of all the linear (resp., positive, affine, convex) combinations of the vectors in S.

Let $P, C \subseteq \mathbb{R}^n$, where $P \cup C = S$. We denote by $\operatorname{nnc.hull}(P, C)$ the set of all convex combinations of the vectors in S such that $\lambda_j > 0$ for some $\boldsymbol{x}_j \in P$ (informally, we say that there exists a vector of P that plays an active role in the convex combination). Note that $\operatorname{nnc.hull}(P, C) = \operatorname{nnc.hull}(P, P \cup C)$ so that, if $C \subseteq P$,

$$\operatorname{convex.hull}(P) = \operatorname{nnc.hull}(P, \varnothing) = \operatorname{nnc.hull}(P, P) = \operatorname{nnc.hull}(P, C).$$

It can be observed that $\operatorname{linear.hull}(S)$ is an affine space, $\operatorname{conic.hull}(S)$ is a topologically closed convex cone, $\operatorname{convex.hull}(S)$ is a topologically closed polytope, and $\operatorname{nnc.hull}(P, C)$ is an NNC polytope.

Points, Closure Points, Rays and Lines

Let $\mathcal{P} \in \mathbb{P}_n$ be an NNC polyhedron. Then

- a vector $\boldsymbol{p} \in \mathcal{P}$ is called a *point* of $\mathcal{P}$;
- a vector $\boldsymbol{c} \in \mathbb{R}^n$ is called a *closure point* of $\mathcal{P}$ if it is a point of the topological closure of $\mathcal{P}$;
- a vector $\boldsymbol{r} \in \mathbb{R}^n$, where $\boldsymbol{r} \neq \boldsymbol{0}$, is called a *ray* (or direction of infinity) of $\mathcal{P}$ if $\mathcal{P} \neq \varnothing$ and $\boldsymbol{p} + \lambda\boldsymbol{r} \in \mathcal{P}$, for all points $\boldsymbol{p} \in \mathcal{P}$ and all $\lambda \in \mathbb{R}_+$;
- a vector $\boldsymbol{l} \in \mathbb{R}^n$ is called a *line* of $\mathcal{P}$ if both $\boldsymbol{l}$ and $-\boldsymbol{l}$ are rays of $\mathcal{P}$.

A point of an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ is a *vertex* if and only if it cannot be expressed as a convex combination of any other pair of distinct points in $\mathcal{P}$. A ray $\boldsymbol{r}$ of a polyhedron $\mathcal{P}$ is an *extreme ray* if and only if it cannot be expressed as a positive combination of any other pair $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ of rays of $\mathcal{P}$, where $\boldsymbol{r} \neq \lambda\boldsymbol{r}_1$, $\boldsymbol{r} \neq \lambda\boldsymbol{r}_2$ and $\boldsymbol{r}_1 \neq \lambda\boldsymbol{r}_2$ for all $\lambda \in \mathbb{R}_+$ (i.e., rays differing by a positive scalar factor are considered to be the same ray).

Generators Representation

Each NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ can be represented by finite sets of lines L, rays R, points P and closure points C of $\mathcal{P}$. The 4-tuple $\mathcal{G} = (L, R, P, C)$ is said to be a *generator system* for $\mathcal{P}$, in the sense that

$$\mathcal{P} = \operatorname{linear.hull}(L) + \operatorname{conic.hull}(R) + \operatorname{nnc.hull}(P, C),$$

where the symbol '$+$' denotes the Minkowski's sum.

When $\mathcal{P} \in \mathbb{CP}_n$ is a closed polyhedron, then it can be represented by finite sets of lines L, rays R and points P of $\mathcal{P}$. In this case, the 3-tuple $\mathcal{G} = (L, R, P)$ is said to be a *generator system* for $\mathcal{P}$ since we have

$$\mathcal{P} = \mathrm{linear.hull}(L) + \mathrm{conic.hull}(R) + \mathrm{convex.hull}(P).$$

Thus, in this case, every closure point of $\mathcal{P}$ is a point of $\mathcal{P}$.

For any $\mathcal{P} \in \mathbb{P}_n$ and generator system $\mathcal{G} = (L, R, P, C)$ for $\mathcal{P}$, we have $\mathcal{P} = \varnothing$ if and only if $P = \varnothing$. Also P must contain all the vertices of $\mathcal{P}$ although $\mathcal{P}$ can be non-empty and have no vertices. In this case, as P is necessarily non-empty, it must contain points of $\mathcal{P}$ that are *not* vertices. For instance, the half-space of $\mathbb{R}^2$ corresponding to the single constraint $y \geq 0$ can be represented by the generator system $\mathcal{G} = (L, R, P, C)$ such that $L = \{(1,0)^{\mathrm{T}}\}$, $R = \{(0,1)^{\mathrm{T}}\}$, $P = \{(0,0)^{\mathrm{T}}\}$, and $C = \varnothing$. It is also worth noting that the only ray in R is *not* an extreme ray of $\mathcal{P}$.

Minimized Representations

A constraints system $\mathcal{C}$ for an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ is said to be *minimized* if no proper subset of $\mathcal{C}$ is a constraint system for $\mathcal{P}$.

Similarly, a generator system $\mathcal{G} = (L, R, P, C)$ for an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ is said to be *minimized* if there does not exist a generator system $\mathcal{G}' = (L', R', P', C') \neq \mathcal{G}$ for $\mathcal{P}$ such that $L' \subseteq L$, $R' \subseteq R$, $P' \subseteq P$ and $C' \subseteq C$.

Double Description

Any NNC polyhedron $\mathcal{P}$ can be described by using a constraint system $\mathcal{C}$, a generator system $\mathcal{G}$, or both by means of the *double description pair (DD pair)* $(\mathcal{C}, \mathcal{G})$. The *double description method* is a collection of well-known as well as novel theoretical results showing that, given one kind of representation, there are algorithms for computing a representation of the other kind and for minimizing both representations by removing redundant constraints/generators.

Such changes of representation form a key step in the implementation of many operators on NNC polyhedra: this is because some operators, such as intersections and poly-hulls, are provided with a natural and efficient implementation when using one of the representations in a DD pair, while being rather cumbersome when using the other.

Topologies and Topological-compatibility

As indicated above, when an NNC polyhedron $\mathcal{P}$ is necessarily closed, we can ignore the closure points contained in its generator system $\mathcal{G} = (L, R, P, C)$ (as every closure point is also a point) and represent $\mathcal{P}$ by the triple (L, R, P). Similarly, $\mathcal{P}$ can be represented by a constraint system that has no strict inequalities. Thus a necessarily closed polyhedron can have a smaller representation than one that is not necessarily closed. Moreover, operators restricted to work on closed polyhedra only can be implemented more efficiently. For this reason the library provides two alternative “topological kinds” for a polyhedron, *NNC* and *C*. We shall abuse terminology by referring to the topologcal kind of a polyhedron as its *topology*.

In the library, the topology of each polyhedron object is fixed once for all at the time of its creation and must be respected when performing operations on the polyhedron.

Unless it is otherwise stated, all the polyhedra, constraints and/or generators in any library operation must obey the following *topological-compatibility* rules:

- polyhedra are topologically-compatible if and only if they have the same topology;
- all constraints except for strict inequality constraints and all generators except for closure points are topologically-compatible with both C and NNC polyhedra;
- strict inequality constraints and closure points are topologically-compatible with a polyhedron if and only if it is NNC.

Wherever possible, the library provides methods that, starting from a polyhedron of a given topology, build the corresponding polyhedron having the other topology.

Space Dimensions and Dimension-compatibility

The *space dimension* of an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ (resp., a C polyhedron $\mathcal{P} \in \mathbb{CP}_n$) is the dimension $n \in \mathbb{N}$ of the corresponding vector space $\mathbb{R}^n$. The space dimension of constraints, generators and other objects of the library is defined similarly.

Unless it is otherwise stated, all the polyhedra, constraints and/or generators in any library operation must obey the following space *dimension-compatibility* rules:

- polyhedra are dimension-compatible if and only if they have the same space dimension;
- the constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b$ where $\bowtie \in \{=, \geq, >\}$ and $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^m$, is dimension-compatible with a polyhedron having space dimension n if and only if $m \leq n$;
- the generator $\boldsymbol{x} \in \mathbb{R}^m$ is dimension-compatible with a polyhedron having space dimension n if and only if $m \leq n$;
- a system of constraints (resp., generators) is dimension-compatible with a polyhedron if and only if all the constraints (resp., generators) in the system are dimension-compatible with the polyhedron.

While the space dimension of a constraint, a generator or a system thereof is automatically adjusted when needed, the space dimension of a polyhedron can only be changed by explicit calls to operators provided for that purpose.

Rational Polyhedra

An NNC polyhedron is called *rational* if it can be represented by a constraint system where all the constraints have rational coefficients. It has been shown that an NNC polyhedron is rational if and only if it can be represented by a generator system where all the generators have rational coefficients.

The library only supports rational polyhedra. The restriction to rational numbers applies not only to polyhedra, but also to the other numeric arguments that may be required by the operators considered, such as the coefficients defining (rational) affine transformations and (rational) bounding boxes.

1.4 Operations on Convex Polyhedra

In this section we briefly describe operations on NNC polyhedra that are provided by the library.

Intersection and Convex Polyhedral Hull

For any pair of NNC polyhedra $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{P}_n$, the *intersection* of $\mathcal{P}_1$ and $\mathcal{P}_2$, defined as the set intersection $\mathcal{P}_1 \cap \mathcal{P}_2$, is the biggest NNC polyhedron included in both $\mathcal{P}_1$ and $\mathcal{P}_2$; similarly, the *convex polyhedral hull* (or *poly-hull*) of $\mathcal{P}_1$ and $\mathcal{P}_2$, denoted by $\mathcal{P}_1 \uplus \mathcal{P}_2$, is the smallest NNC polyhedron that includes both $\mathcal{P}_1$ and $\mathcal{P}_2$. The intersection and poly-hull of any pair of closed polyhedra in $\mathbb{CP}_n$ is also closed.

In theoretical terms, the intersection and poly-hull operators defined above are the binary *meet* and the binary *join* operators on the lattices $\mathbb{P}_n$ and $\mathbb{CP}_n$.

Convex Polyhedral Difference

For any pair of NNC polyhedra $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{P}_n$, the *convex polyhedral difference* (or *poly-difference*) of $\mathcal{P}_1$ and $\mathcal{P}_2$ is defined as the smallest convex polyhedron containing the set-theoretic difference of $\mathcal{P}_1$ and $\mathcal{P}_2$.

In general, even though $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{CP}_n$ are topologically closed polyhedra, their poly-difference may be a convex polyhedron that is not topologically closed. For this reason, when computing the poly-difference of two C polyhedra, the library will enforce the topological closure of the result.

Concatenating Polyhedra

Viewing a polyhedron as a set of tuples (its points), it is sometimes useful to consider the set of tuples obtained by concatenating an ordered pair of polyhedra. Formally, the *concatenation* of the polyhedra

$\mathcal{P} \in \mathbb{P}_n$ and $\mathcal{Q} \in \mathbb{P}_m$ (taken in this order) is the polyhedron $\mathcal{R} \in \mathbb{P}_{n+m}$ such that

$$\mathcal{R} \stackrel{\text{def}}{=} \Bigl\{ (x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1})^{\mathrm{T}} \in \mathbb{R}^{n+m} \Bigm| (x_0, \ldots, x_{n-1})^{\mathrm{T}} \in \mathcal{P}, (y_0, \ldots, y_{m-1})^{\mathrm{T}} \in \mathcal{Q} \Bigr\}.$$

Another way of seeing it is as follows: first embed polyhedron $\mathcal{P}$ into a vector space of dimension $n + m$ and then add a suitably renamed-apart version of the constraints defining $\mathcal{Q}$.

Adding New Dimensions to the Vector Space

The library provides two operators for adding a number i of space dimensions to an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$, therefore transforming it into a new NNC polyhedron $\mathcal{Q} \in \mathbb{P}_{n+i}$. In both cases, the added dimensions of the vector space are those having the highest indices.

The operator `add_dimensions_and_embed` *embeds* the polyhedron $\mathcal{P}$ into the new vector space of dimension $i + n$ and returns the polyhedron $\mathcal{Q}$ defined by all and only the constraints defining $\mathcal{P}$ (the variables corresponding to the added dimensions are unconstrained). For instance, when starting from a polyhedron $\mathcal{P} \subseteq \mathbb{R}^2$ and adding a third dimension, the result will be the polyhedron

$$\mathcal{Q} = \bigl\{ (x_0, x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^3 \bigm| (x_0, x_1)^{\mathrm{T}} \in \mathcal{P} \bigr\}.$$

In contrast, the operator `add_dimensions_and_project` *projects* the polyhedron $\mathcal{P}$ into the new vector space of dimension $i + n$ and returns the polyhedron $\mathcal{Q}$ whose constraint system, besides the constraints defining $\mathcal{P}$, will include additional constraints on the added dimensions. Namely, the corresponding variables are all constrained to be equal to 0. For instance, when starting from a polyhedron $\mathcal{P} \subseteq \mathbb{R}^2$ and adding a third dimension, the result will be the polyhedron

$$\mathcal{Q} = \bigl\{ (x_0, x_1, 0)^{\mathrm{T}} \in \mathbb{R}^3 \bigm| (x_0, x_1)^{\mathrm{T}} \in \mathcal{P} \bigr\}.$$

Removing Dimensions from the Vector Space

The library provides two operators for removing space dimensions from an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$, therefore transforming it into a new NNC polyhedron $\mathcal{Q} \in \mathbb{P}_m$ where $m \leq n$.

Given a set of variables, the operator `remove_dimensions` removes all the space dimensions specified by the variables in the set. For instance, letting $\mathcal{P} \in \mathbb{P}_4$ be the singleton set $\bigl\{(3, 1, 0, 2)^{\mathrm{T}}\bigr\} \subseteq \mathbb{R}^4$, then after invoking this operator with the set of variables $\{x_1, x_2\}$ the resulting polyhedron is

$$\mathcal{Q} = \bigl\{(3, 2)^{\mathrm{T}}\bigr\} \subseteq \mathbb{R}^2.$$

Given a space dimension m less than or equal to that of the polyhedron, the operator `remove_higher_dimensions` removes the dimensions having indices greater than or equal to m. For instance, letting $\mathcal{P} \in \mathbb{P}_4$ defined as before, by invoking this operator with $m = 2$ the resulting polyhedron will be

$$\mathcal{Q} = \bigl\{(3, 1)^{\mathrm{T}}\bigr\} \subseteq \mathbb{R}^2.$$

Mapping the Dimensions of the Vector Space

The operator `map_dimensions` provided by the library maps the dimensions of the vector space $\mathbb{R}^n$ according to a partial injective function $\rho \colon \{0, \ldots, n-1\} \rightarrowtail \mathbb{N}$ such that $\rho\bigl(\{0, \ldots, n-1\}\bigr) = \{0, \ldots, m-1\}$ with $m \leq n$. Dimensions corresponding to indices that are not mapped by ρ are removed.

If $m = 0$, i.e., if the function ρ is undefined everywhere, then the operator projects the argument polyhedron $\mathcal{P} \in \mathbb{P}_n$ onto the zero-dimension space $\mathbb{R}^0$; otherwise the result is $\mathcal{Q} \in \mathbb{P}_m$ given by

$$\mathcal{Q} \stackrel{\text{def}}{=} \Bigl\{ \bigl(v_{\rho^{-1}(0)}, \ldots, v_{\rho^{-1}(m-1)}\bigr)^{\mathrm{T}} \Bigm| (v_0, \ldots, v_{n-1})^{\mathrm{T}} \in \mathcal{P} \Bigr\}.$$

Expanding One Dimension of the Vector Space to Multiple Dimensions

The operator `expand_dimension` provided by the library adds m new dimensions to a polyhedron $\mathcal{P} \in \mathbb{P}_n$, with $n > 0$, so that dimensions n, $n+1$, ..., $n+m-1$ of the result $\mathcal{Q}$ are exact copies of the i-th dimension of $\mathcal{P}$. More formally,

$$\mathcal{Q} \stackrel{\text{def}}{=} \left\{ \boldsymbol{u} \in \mathbb{R}^{n+m} \;\middle|\; \begin{array}{l} \exists \boldsymbol{v}, \boldsymbol{w} \in \mathcal{P} \mathrel{.} u_i = v_i \\ \qquad \wedge \forall j = n, n+1, \ldots, n+m-1 : u_j = w_i \\ \qquad \wedge \forall k = 0, \ldots, n-1 : k \neq i \implies u_k = v_k = w_k \end{array} \right\}.$$

This operation has been proposed in **[GDMDRS04]**(p. 16).

Folding Multiple Dimensions of the Vector Space into One Dimension

The operator `fold_dimensions` provided by the library, given a polyhedron $\mathcal{P} \in \mathbb{P}_n$, with $n > 0$, folds a set of dimensions $J = \{j_0, \ldots, j_{m-1}\}$, with $m < n$ and $j < n$ for each $j \in J$, into dimension $i < n$, where $i \notin J$. The result is given by

$$\mathcal{Q} \stackrel{\text{def}}{=} \biguplus_{d=0}^{m} \mathcal{Q}_d$$

where

$$\mathcal{Q}_m \stackrel{\text{def}}{=} \left\{ \boldsymbol{u} \in \mathbb{R}^{n-m} \;\middle|\; \begin{array}{l} \exists \boldsymbol{v} \in \mathcal{P} \mathrel{.} u_{i'} = v_i \\ \qquad \wedge \forall k = 0, \ldots, n-1 : k \neq i \implies u_{k'} = v_k \end{array} \right\}$$

and, for $d = 0$, ..., $m-1$,

$$\mathcal{Q}_d \stackrel{\text{def}}{=} \left\{ \boldsymbol{u} \in \mathbb{R}^{n-m} \;\middle|\; \begin{array}{l} \exists \boldsymbol{v} \in \mathcal{P} \mathrel{.} u_{i'} = v_{j_d} \\ \qquad \wedge \forall k = 0, \ldots, n-1 : k \neq i \implies u_{k'} = v_k \end{array} \right\},$$

and, finally, for $k = 0$, ..., $n-1$,

$$k' \stackrel{\text{def}}{=} k - \#\{\, j \in J \mid k > j \,\},$$

($\# S$ denotes the cardinality of the finite set S).

This operation has been proposed in **[GDMDRS04]**(p. 16).

Affine Images and Preimages

For each function mapping $\phi \colon \mathbb{R}^n \to \mathbb{R}^m$, we denote by $\phi(S) \subseteq \mathbb{R}^m$ the *image* under ϕ of the set $S \subseteq \mathbb{R}^n$; formally,

$$\phi(S) \stackrel{\text{def}}{=} \left\{ \phi(\boldsymbol{v}) \in \mathbb{R}^m \;\middle|\; \boldsymbol{v} \in S \right\}.$$

Similarly, we denote by $\phi^{-1}(S') \subseteq \mathbb{R}^n$ the *preimage* under ϕ of $S' \subseteq \mathbb{R}^m$, that is the largest set $S \subseteq \mathbb{R}^n$ such that $\phi(S) \subseteq S'$; formally,

$$\phi^{-1}(S') \stackrel{\text{def}}{=} \left\{ \boldsymbol{v} \in \mathbb{R}^n \;\middle|\; \phi(\boldsymbol{v}) \in S' \right\}.$$

The function mapping $\phi \colon \mathbb{R}^n \to \mathbb{R}^m$ is an *affine transformation* if there exist a matrix $A \in \mathbb{R}^m \times \mathbb{R}^n$ and a vector $\boldsymbol{b} \in \mathbb{R}^m$ such that, for all $\boldsymbol{x} \in \mathbb{R}^n$, we have $\phi(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$. If $n = m$, then the function ϕ is said to be *space-dimension preserving*.

Both $\mathbb{P}_n$ and $\mathbb{CP}_n$ are closed under the application of any space-dimension preserving affine image and preimage operators.

The library provides two operators, one computes an affine image and the other an affine preimage of a polyhedron $\mathcal{P} \in \mathbb{P}_n$ for a given variable x_k and linear expression $\mathrm{expr} = \sum_{i=0}^{n-1} a_i x_i + b$. This variable and expression determine the affine transformation ϕ that is to be used by the operator. That is, ϕ is the transformation defined by the matrix and vector

$$A = \begin{pmatrix} 1 & & 0 & 0 & \cdots & \cdots & 0 \\ & \ddots & & \vdots & & & \vdots \\ 0 & & 1 & 0 & \cdots & \cdots & 0 \\ a_0 & \cdots & a_{k-1} & a_k & a_{k+1} & \cdots & a_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 & & 0 \\ \vdots & & & \vdots & & \ddots & \\ 0 & \cdots & \cdots & 0 & 0 & & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the a_i (resp., b) occurs in the $(k+1)$st row in A (resp., position in $\boldsymbol{b}$). Thus ϕ transforms any point $(x_0, \dots, x_{n-1})^{\mathrm{T}}$ in the polyhedron $\mathcal{P}$ to

$$\Bigl(x_0, \dots, \bigl(\textstyle\sum_{i=0}^{n-1} a_i x_i + b\bigr), \dots, x_{n-1}\Bigr)^{\mathrm{T}}.$$

The affine image operator computes the affine image of $\mathcal{P}$ under ϕ. For instance, suppose the polyhedron $\mathcal{P}$ to be transformed is the square in $\mathbb{R}^2$ generated by the set of points $\bigl\{(0,0)^{\mathrm{T}}, (0,3)^{\mathrm{T}}, (3,0)^{\mathrm{T}}, (3,3)^{\mathrm{T}}\bigr\}$. Then, for example if the considered variable is x_0 and the linear expression $x_0 + 2x_1 + 4$ (so that $k = 0$, $a_0 = 1, a_1 = 2, b = 4$), the affine image operator will translate $\mathcal{P}$ to the parallelogram $\mathcal{P}_1$ generated by the set of points $\bigl\{(4,0)^{\mathrm{T}}, (10,3)^{\mathrm{T}}, (7,0)^{\mathrm{T}}, (13,3)^{\mathrm{T}}\bigr\}$ with height equal to the side of the square and oblique sides parallel to the line $x_0 - 2x_1$. If the considered variable is as before (i.e., $k = 0$) but the linear expression is x_1 (so that $a_0 = 0, a_1 = 1, b = 0$), then the resulting polyhedron $\mathcal{P}_2$ is the positive diagonal of the square.

The affine preimage operator computes the affine preimage of $\mathcal{P}$ under ϕ. For instance, suppose now that we apply the affine preimage operator as given in the first example using variable x_0 and linear expression $x_0 + 2x_1 + 4$ to the parallelogram $\mathcal{P}_1$; then we get the original square $\mathcal{P}$ back. If, on the other hand, we apply the affine preimage operator as given in the second example using variable x_0 and linear expression x_1 to $\mathcal{P}_2$, then the resulting polyhedron is a line that corresponds to the x_1 axes.

Observe that provided the coefficient a_k of the considered variable in the linear expression is non-zero, the affine transformation is invertible.

Generalized Affine Images

The library provides another operator which is a generalization of the affine image operator. Given a polyhedron $\mathcal{P} \in \mathbb{P}_n$, an affine expression $\mathrm{lhs} = \sum_{i=0}^{n-1} a'_i x_i + b'$, a relation symbol $\bowtie \in \{<, \leq, =, \geq, >\}$, and an affine expression $\mathrm{rhs} = \sum_{i=0}^{n-1} a_i x_i + b$, the image of $\mathcal{P}$ with respect to the transfer function $\mathrm{lhs} \bowtie \mathrm{rhs}$ is defined as

$$\left\{ (w_0, \dots, w_{n-1})^{\mathrm{T}} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} (v_0, \dots, v_{n-1})^{\mathrm{T}} \in \mathcal{P}, \\ \bigl(i \in \{0, \dots, n-1\} \land a'_i = 0 \implies w_i = v_i\bigr), \\ \sum_{i=0}^{n-1} a'_i w_i + b' \bowtie \sum_{i=0}^{n-1} a_i v_i + b \end{array} \right\}.$$

Note that, when $\mathrm{lhs} = x_k$ and $\bowtie \in \{=\}$, then the above operator is equivalent to the application of the standard affine image of $\mathcal{P}$ with respect to the variable x_k and the affine expression rhs (hence the name given to this operator).

Time-Elapse Operator

The *time-elapse* operator has been defined in **[HPR97]**(p. 17). Actually, the time-elapse operator provided by the library is a slight generalization of that one, since it also works on NNC polyhedra. For any two NNC polyhedra $\mathcal{P}, \mathcal{Q} \in \mathbb{P}_n$, the time-elapse between $\mathcal{P}$ and $\mathcal{Q}$, denoted $\mathcal{P} \nearrow \mathcal{Q}$, is the smallest NNC polyhedron containing the set

$$\bigl\{\, \boldsymbol{p} + \lambda \boldsymbol{q} \in \mathbb{R}^n \bigm| \boldsymbol{p} \in \mathcal{P}, \boldsymbol{q} \in \mathcal{Q}, \lambda \in \mathbb{R}_+ \,\bigr\}.$$

Note that, if $\mathcal{P}, \mathcal{Q} \in \mathbb{CP}_n$ are closed polyhedra, the above set is also a closed polyhedron. In contrast, when $\mathcal{Q}$ is not topologically closed, the above set might not be an NNC polyhedron.

Relation-with Operators

The library provides operators for checking the relation holding between an NNC polyhedron and either a constraint or a generator.

Suppose $\mathcal{P}$ is an NNC polyhedron and $\mathcal{C}$ an arbitrary constraint system representing $\mathcal{P}$. Suppose also that $c = \bigl(\langle \boldsymbol{a}, \boldsymbol{x} \rangle \bowtie b\bigr)$ is a constraint with $\bowtie \in \{=, \geq, >\}$ and $\mathcal{Q}$ the set of points that satisfy c. The possible relations between $\mathcal{P}$ and c are as follows.

- $\mathcal{P}$ *is disjoint* from c if $\mathcal{P} \cap \mathcal{Q} = \varnothing$; that is, adding c to $\mathcal{C}$ gives us the empty polyhedron.
- $\mathcal{P}$ *strictly intersects* c if $\mathcal{P} \cap \mathcal{Q} \neq \varnothing$ and $\mathcal{P} \cap \mathcal{Q} \subset \mathcal{P}$; that is, adding c to $\mathcal{C}$ gives us a non-empty polyhedron strictly smaller than $\mathcal{P}$.
- $\mathcal{P}$ *is included* in c if $\mathcal{P} \subseteq \mathcal{Q}$; that is, adding c to $\mathcal{C}$ leaves $\mathcal{P}$ unchanged.
- $\mathcal{P}$ *saturates* c if $\mathcal{P} \subseteq \mathcal{H}$, where $\mathcal{H}$ is the hyperplane induced by constraint c, i.e., the set of points satisfying the equality constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = b$; that is, adding the constraint $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = b$ to $\mathcal{C}$ leaves $\mathcal{P}$ unchanged.

The polyhedron $\mathcal{P}$ *subsumes* the generator g if adding g to any generator system representing $\mathcal{P}$ does not change $\mathcal{P}$.

Intervals, boxes and bounding boxes

An *interval* in $\mathbb{R}$ is a pair of *bounds*, called *lower* and *upper*. Each bound can be either (1) *closed and bounded*, (2) *open and bounded*, or (3) *open and unbounded*. If the bound is *bounded*, then it has a value in $\mathbb{R}$. An n-dimensional *box* $\mathcal{B}$ in $\mathbb{R}^n$ is a sequence of n intervals in $\mathbb{R}$.

The polyhedron $\mathcal{P}$ *represents a box* $\mathcal{B}$ in $\mathbb{R}^n$ if $\mathcal{P}$ is described by a constraint system in $\mathbb{R}^n$ that consists of one constraint for each bounded bound (lower and upper) in an interval in $\mathcal{B}$: Letting $\boldsymbol{e}_i = (0, \dots, 1, \dots, 0)^{\mathrm{T}}$ be the vector in $\mathbb{R}^n$ with 1 in the i'th position and zeroes in every other position; if the lower bound of the i'th interval in $\mathcal{B}$ is bounded, the corresponding constraint is defined as $\langle \boldsymbol{e}_i, \boldsymbol{x} \rangle \bowtie b$, where b is the value of the bound and $\bowtie$ is $\geq$ if it is a closed bound and $>$ if it is an open bound. Similarly, if the upper bound of the i'th interval in $\mathcal{B}$ is bounded, the corresponding constraint is defined as $\langle \boldsymbol{e}_i, \boldsymbol{x} \rangle \bowtie b$, where b is the value of the bound and $\bowtie$ is $\leq$ if it is a closed bound and $<$ if it is an open bound.

If every bound in the intervals defining a box $\mathcal{B}$ is either closed and bounded or open and unbounded, then $\mathcal{B}$ represents a closed polyhedron.

The *bounding box* of an NNC polyhedron $\mathcal{P}$ is the smallest n-dimensional box containing $\mathcal{P}$.

The library provides operations for computing the bounding box of an NNC polyhedron and conversely, for obtaining the NNC polyhedron representing a given bounding box.

Widening Operators

The library provides two widening operators for the domain of NNC polyhedra. The first one, that we call *H79-widening*, mainly follows the specification provided in the PhD thesis of N. Halbwachs **[Hal79]**(p. 16), also described in **[HPR97]**(p. 17). There are a few differences between the H79-widening and the widening described in the cited paper. In particular, the H79-widening of an NNC polyhedron $\mathcal{P} \in \mathbb{P}_n$ using the NNC polyhedron $\mathcal{Q} \in \mathbb{P}_n$:

- allows for equalities in $\mathcal{P}$ and $\mathcal{Q}$ (the original definition is restricted to inequalities);
- requires as a precondition that $\mathcal{Q} \subseteq \mathcal{P}$.

The second widening operator, that we call *BHRZ03-widening*, is an instance of the specification provided in **[BHRZ03a]**(p. 15). This operator also requires as a precondition that $\mathcal{Q} \subseteq \mathcal{P}$ and it is guaranteed to provide a result which is at least as precise as the H79-widening.

Both widening operators can be applied to polyhedra that are not topologically closed. The user is warned that, in such a case, the results may not closely match the geometric intuition which is at the base of the specification of the two widenings. The reason is that, in the current implementation, the widenings are not directly applied to the NNC polyhedra, but rather to their internal representations. Implementation work is in progress and future versions of the library may provide an even better integration of the two widenings with the domain of NNC polyhedra.

Widening with Tokens

When approximating a fixpoint computation using widening operators, a common tactic to improve the precision of the final result is to delay the application of widening operators. The usual approach is to fix a parameter k and only apply widenings starting from the k-th iteration.

The library also supports an improved widening delay strategy, that we call *widening with tokens* **[BHRZ03a]**(p. 15). A token is a sort of wildcard allowing for the replacement of the widening application by the exact upper bound computation: the token is used (and thus consumed) only when the widening would have resulted in an actual precision loss (as opposed to the *potential* precision loss of the classical delay strategy). Thus, all widening operators can be supplied with an optional argument, recording the number of available tokens, which is decremented when tokens are used. The approximated fixpoint computation will start with a fixed number k of tokens, which will be used if and when needed. When there are no tokens left, the widening is always applied.

Extrapolation Operators

Besides the two widening operators, the library also implements several *extrapolation* operators, which differ from widenings in that their use along an upper iteration sequence does not ensure convergence in a finite number of steps.

In particular, for each of the two widenings there is a corresponding *limited* extrapolation operator, which can be used to implement the *widening "up to"* technique as described in **[HPR97]**(p. 17). Each limited extrapolation operator takes a constraint system as an additional parameter and uses it to improve the approximation yielded by the corresponding widening operator. Note that a convergence guarantee can only be obtained by suitably restricting the set of constraints that can occur in this additional parameter. For instance, in **[HPR97]**(p. 17) this set is fixed once and for all before starting the computation of the upward iteration sequence.

The *bounded* extrapolation operators further enhance each one of the limited extrapolation operators described above, by ensuring that their results cannot be worse than the smallest **bounding box**(p. 10) enclosing the two argument polyhedra.

1.5 The Powerset Construction

The PPL provides the finite powerset construction; this takes a pre-existing domain and upgrades it to one that can represent disjunctive information (by using a *finite* number of disjucts). The construction follows the approach described in **[Bag98]**(p. 15), also summarised in **[BHZ04]**(p. 15) where there is an account of generic widenings for the powerset domain (some of which are supported in the instantiation of this construction by the domain of convex polyhedra and described in Section **The Polyhedra Powerset Domain**(p. 12)).

The Powerset Domain

The domain is built from a pre-existing base-level domain D which must include an entailment relation '$\vdash$', a meet operation '$\otimes$', a top element '**1**' and bottom element '**0**'.

As the intended semantics of an element of the powerset of the base-level domain is that of disjunction, elements of the powerset are always *reduced* to semantically-equivalent non-redundant elements.

A set $\mathcal{S} \in \wp(D)$ is called *non-redundant* with respect to '$\vdash$' if and only if $\mathbf{0} \notin \mathcal{S}$ and $\forall d_1, d_2 \in \mathcal{S} : d_1 \vdash d_2 \implies d_1 = d_2$. The set of finite non-redundant subsets of D (with respect to '$\vdash$') is denoted by $\wp^{\vdash}_{\mathrm{fn}}(D)$. The reduction function $\Omega^{\vdash}_D \colon \wp_{\mathrm{f}}(D) \to \wp^{\vdash}_{\mathrm{fn}}(D)$ mapping a finite set into its non-redundant counterpart is defined, for each $\mathcal{S} \in \wp_{\mathrm{f}}(D)$, by

$$\Omega^{\vdash}_D(\mathcal{S}) \stackrel{\mathrm{def}}{=} \mathcal{S} \setminus \{\, d \in \mathcal{S} \mid d = \mathbf{0} \text{ or } \exists d' \in \mathcal{S} \,.\, d \Vdash d' \,\}.$$

The restriction to the finite subsets reflects the fact that here disjunctions are implemented by explicit collections of elements of the base-level abstract domain. As a consequence of this restriction, for any $\mathcal{S} \in \wp_{\mathrm{f}}(D)$ such that $\mathcal{S} \neq \{\mathbf{0}\}$, $\Omega^{\vdash}_D(\mathcal{S})$ is the (finite) set of the maximal elements of $\mathcal{S}$.

The *finite powerset domain* over a domain D is the set of all finite reduced sets of D and denoted by D_{P}. The domain includes an approximation ordering '$\vdash_{\mathrm{P}}$' defined so that $\mathcal{S}_1 \vdash_{\mathrm{P}} \mathcal{S}_2$ if and only if

$$\forall d_1 \in \mathcal{S}_1 : \exists d_2 \in \mathcal{S}_2 \,.\, d_1 \vdash d_2.$$

Therefore the top element is $\{\mathbf{1}\}$ and the bottom element is the emptyset.

1.6 Operations on the Powerset Construction

In this section we briefly describe the generic operations on Powerset Domains that are provided by the library for any given base-level domain D.

Meet and Upper Bound

Given the sets $\mathcal{S}_1$ and $\mathcal{S}_2 \in D_{\mathrm{P}}$, the *meet* and *upper bound* operators provided by the library returns the set $\Omega^{\vdash}_D\bigl(\{d_1 \otimes d_2 \mid d_1 \in \mathcal{S}_1, d_2 \in \mathcal{S}_2\}\bigr)$ and reduced set union $\Omega^{\vdash}_D(\mathcal{S}_1 \cup \mathcal{S}_2)$ respectively.

Adding a Disjunct

Given the powerset element $\mathcal{S} \in D_{\mathrm{P}}$ and the base-level element $d \in D$, the *add_disjunct* operator provided by the library returns the powerset element $\Omega^{\vdash}_D\bigl(\mathcal{S} \cup \{d\}\bigr)$.

Collapsing a Powerset Element

If the given powerset element is not empty, then the *collapse* operator returns the singleton powerset consisting of an upper-bound of all the disjuncts.

1.7 The Polyhedra Powerset Domain

The Polyhedra powerset domain $(\mathbb{P}_n)_{\mathrm{P}}$ provided by the PPL is the finite powerset domain (defined in Section **The Powerset Construction**(p. 11)) over the domain of NNC polyhedra $\mathbb{P}_n$.

In addition to the operations described for the generic powerset domain in Section **Operations on the Powerset Construction**(p. 12), we provide some operations that are specific to this instantiation. Of these, most correspond to the application of the equivalent operation on each of the NNC polyhedron that are in the given set. Here we just describe those operations that are particular to the polyhedra powerset domain.

Geometric Comparisons

Given the sets $\mathcal{S}_1, \mathcal{S}_2 \in (\mathbb{P}_n)_{\mathrm{P}}$, then we say that $\mathcal{S}_1$ *geometrically_covers* $\mathcal{S}_2$ if every point (in some element) in a polyhedron in $\mathcal{S}_2$ is also a point in a polyhedron in $\mathcal{S}_1$. If $\mathcal{S}_1$ geometrically_covers $\mathcal{S}_2$ and $\mathcal{S}_2$ geometrically_covers $\mathcal{S}_1$, then we say that they are *geometrically_equal*.

Pairwise Merge

Given the powerset $\mathcal{S} \in (\mathbb{P}_n)_{\mathrm{P}}$, then the *pairwise_merge* operator takes pairs of distinct elements in $\mathcal{S}$ whose poly-hull is the same as their set-theoretical union and replaces them by their union. This replacement is done recursively so that, for each pair $\mathcal{P}, \mathcal{Q}$ of distinct polyhedra in the result set, we have $\mathcal{P} \uplus \mathcal{Q} \neq \mathcal{P} \cup \mathcal{Q}$.

Extrapolation Operators

The library implements a generalization of the extrapolation operator for powerset domains proposed in **[BGP99]**(p. 15). The operator `BGP99_extrapolation_assign` is made parametric by allowing for the specification of a base-level extrapolation operator different from the H79 widening (e.g., the BHRZ03 widening can be used). Note that, in the general case, this operator cannot guarantee the convergence of the iteration sequence in a finite number of steps (for a counter-example, see **[BHZ04]**(p. 15)).

Certificate-Based Widenings

The PPL library provides support for the specification of proper widening operators on the powerset domain of convex polyhedra. In particular, this version of the library implements an instance of the *certificate-based widening framework* proposed in **[BHZ03b]**(p. 15).

A *finite convergence certificate* for an extrapolation operator is a formal way of ensuring that such an operator is indeed a widening on the considered domain. Given a widening operator on the base-level domain, together with the corresponding convergence certificate, the BHZ03 framework shows how it is possible to lift this widening so as to work on the finite powerset domain, while still ensuring convergence in a finite number of iterations.

Being highly parametric, the BHZ03 widening framework can be instantiated in many ways. The current implementation provides the templatic operator `BHZ03_widening_assign<Certificate, Widening>` which only exploits a fraction of this generality, by allowing the user to specify the base-level widening function and the corresponding certificate. The widening strategy is fixed and uses two extrapolation heuristics: first, the least upper bound is tried; second, the **BGP99 extrapolation operator**(p. 13) is tried, possibly applying **pairwise merging**(p. 12). If both heuristics fail to converge according to the convergence certificate, then an attempt is made to apply the base-level widening to the poly-hulls of the two arguments, possibly improving the result obtained by means of the **poly-difference**(p. 6) operator. For more details and a justification of the overall approach, see **[BHZ03b]**(p. 15) and **[BHZ04]**(p. 15).

The library provides two convergence certificates: while **BHRZ03_Certificate**(p. 70) is compatible with both the BHRZ03 and the H79 widenings, **H79_Certificate**(p. 88) is only compatible with the latter. Note that using different certificates will change the results obtained, even when using the same base-level widening operator. It is also worth stressing that it is up to the user to see that the widening operator is actually compatible with a given convergence certificate. If such a requirement is not met, then an extrapolation operator will be obtained.

1.8 Using the Library

A Note on the Implementation of the Operators

When adopting the double description method for the representation of convex polyhedra, the implementation of most of the operators may require an explicit conversion from one of the two representations into the other one, leading to algorithms having a worst-case exponential complexity. However, thanks to the adoption of lazy and incremental computation techniques, the library turns out to be rather efficient in many practical cases.

In earlier versions of the library, a number of operators were introduced in two flavors: a *lazy* version and an *eager* version, the latter having the operator name ending with `_and_minimize`. In principle, only the lazy versions should be used. The eager versions were added to help a knowledgeble user obtain better performance in particular cases. Basically, by invoking the eager version of an operator, the user is trading laziness to better exploit the incrementality of the inner library computations. Starting from version

0.5, the lazy and incremental computation techniques have been refined to achieve a better integration: as a consequence, the lazy versions of the operators are now almost always more efficient than the eager versions.

One of the cases when an eager computation still makes sense is when the well-known *fail-first* principle comes into play. For instance, if you have to compute the intersection of several polyhedra and you strongly suspect that the result will become empty after a few of these intersections, then you may obtain a better performance by calling the eager version of the intersection operator, since the minimization process also enforces an emptyness check. Note anyway that the same effect can be obtained by interleaving the calls of the lazy operator with explicit emptyness checks.

On Object-Orientation and Polymorphism: A Disclaimer

The PPL library is mainly a collection of so-called “concrete data types”: while providing the user with a clean and friendly interface, these types are not meant to — i.e., they should not — be used polymorphically (since, e.g., most of the destructors are not declared `virtual`). In practice, this restriction means that the library types should not be used as *public base classes* to be derived from. A user willing to extend the library types, adding new functionalities, often can do so by using *containment* instead of inheritance; even when there is the need to override a `protected` method, non-public inheritance should suffice.

On Const-Correctness: A Warning about the Use of References and Iterators

Most operators of the library depend on one or more parameters that are declared “const”, meaning that they will not be changed by the application of the considered operator. Due to the adoption of lazy computation techniques, in many cases such a const-correctness guarantee only holds at the semantic level, whereas it does not necessarily hold at the implementation level. For a typical example, consider the extraction from a polyhedron of its constraint system representation. While this operation is not going to change the polyhedron, it might actually invoke the internal conversion algorithm and modify the generators representation of the polyhedron object, e.g., by reordering the generators and removing those that are detected as redundant. Thus, any previously computed reference to the generators of the polyhedron (be it a direct reference object or an indirect one, such as an iterator) will no longer be valid. For this reason, code fragments such as the following should be avoided, as they may result in undefined behavior:

```
// Find a reference to the first point of the non-empty polyhedron ‘ph’.
const GenSys& gs = ph.generators();
GenSys::const_iterator i = gs.begin();
for (GenSys::const_iterator gs_end = gs.end(); i != gs_end; ++i)
  if (i->is_point())
    break;
const Generator& p = *i;
// Get the constraints of ‘ph’.
const ConSys& cs = ph.constraints();
// Both the const iterator ‘i’ and the reference ‘p’
// are no longer valid at this point.
cout << p.divisor() << endl;  // Undefinded behavior!
++i;                          // Undefinded behavior!
```

As a rule of thumb, if a polyhedron plays any role in a computation (even as a const parameter), then any previously computed reference to parts of the polyhedron may have been invalidated. Note that, in the example above, the computation of the constraint system could have been placed after the uses of the iterator `i` and the reference `p`. Anyway, if really needed, it is always possible to take a copy of, instead of a reference to, the parts of interest of the polyhedron; in the case above, one may have taken a copy of the generator system by replacing the second line of code with the following:

```
GenSys gs = ph.generators();
```

The same observations, modulo syntactic sugar, apply to the operators defined in the C interface of the library.

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2 PPL Module Index

2.1 PPL Modules

Here is a list of all modules:

3 PPL Namespace Index

3.1 PPL Namespace List

Here is a list of all documented namespaces with brief descriptions:

4 PPL Hierarchical Index

4.1 PPL Class Hierarchy

This inheritance list is sorted roughly, but not completely, alphabetically:

5 PPL Class Index

5.1 PPL Class List

Here are the classes, structs, unions and interfaces with brief descriptions:

6 PPL Page Index

6.1 PPL Related Pages

Here is a list of all related documentation pages:

7 PPL Module Documentation

7.1 The Library

The core implementation of the Parma Polyhedra Library is written in C++. See Namespace, Hierarchical and Compound indexes for additional information about each single data type.

7.2 Library Defines

Defines

- #define **PPL_VERSION_MAJOR** 0

The major number of the PPL version.

- #define **PPL_VERSION_MINOR** 6

  The minor number of the PPL version.

- #define **PPL_VERSION_REVISION** 1

  The revision number of the PPL version.

- #define **PPL_VERSION_BETA** 0

  The beta number of the PPL version. This is zero for official releases and nonzero for development snapshots.

- #define **PPL_VERSION** "0.6.1"

  A string containing the PPL version.

7.2.1 Define Documentation

7.2.1.1 #define PPL_VERSION "0.6.1"

A string containing the PPL version.

Let `M` and `m` denote the numbers associated to PPL_VERSION_MAJOR and PPL_VERSION_MINOR, respectively. The format of PPL_VERSION is `M "." m` if both PPL_VERSION_REVISION (`r`) and PPL_VERSION_BETA (`b`)are zero, `M "." m "pre" b` if PPL_VERSION_REVISION is zero and PPL_VERSION_BETA is not zero, `M "." m "." r` if PPL_VERSION_REVISION is not zero and PPL_VERSION_BETA is zero, `M "." m "." r "pre" b` if neither PPL_VERSION_REVISION nor PPL_VERSION_BETA are zero.

7.3 C Language Interface

Some details about the C Interface(p. 40).

Version Checking

- #define **PPL_VERSION_MAJOR** 0

  The major number of the PPL version.

- #define **PPL_VERSION_MINOR** 6

  The minor number of the PPL version.

- #define **PPL_VERSION_REVISION** 1

  The revision number of the PPL version.

- #define **PPL_VERSION_BETA** 0

  The beta number of the PPL version. This is zero for official releases and nonzero for development snapshots.

- #define **PPL_VERSION** "0.6.1"

  A string containing the PPL version.

- int **ppl_version_major** (void)

  Returns the major number of the PPL version.

- int **ppl_version_minor** (void)

  Returns the minor number of the PPL version.

- int **ppl_version_revision** (void)

  Returns the revision number of the PPL version.

- int **ppl_version_beta** (void)

  Returns the beta number of the PPL version.

- int **ppl_version** (const char ∗∗p)

  Writes to `m` *a pointer to a character string containing the PPL version.*

- int **ppl_banner** (const char ∗∗p)

  Writes to `m` *a pointer to a character string containing the PPL banner.*

Simple I/O Functions

- typedef const char ∗ **ppl_io_variable_output_function_type** (**ppl_dimension_type** var)

  The type of output functions used for printing variables.

- int **ppl_io_print_variable** (**ppl_dimension_type** var)

  Pretty-prints `x` *to* `stdout`.

- int **ppl_io_fprint_variable** (FILE ∗stream, **ppl_dimension_type** var)

  Pretty-prints `var` *to the given output* `stream`.

- int **ppl_io_print_Coefficient** (**ppl_const_Coefficient_t** x)

  Prints `x` *to* `stdout`.

- int **ppl_io_fprint_Coefficient** (FILE ∗stream, **ppl_const_Coefficient_t** x)

  Prints `x` *to the given output* `stream`.

- int **ppl_io_print_LinExpression** (**ppl_const_LinExpression_t** x)

  Prints `x` *to* `stdout`.

- int **ppl_io_fprint_LinExpression** (FILE ∗stream, **ppl_const_LinExpression_t** x)

  Prints `x` *to the given output* `stream`.

- int **ppl_io_print_Constraint** (**ppl_const_Constraint_t** x)

  Prints `x` *to* `stdout`.

- int **ppl_io_fprint_Constraint** (FILE ∗stream, **ppl_const_Constraint_t** x)

  Prints `x` *to the given output* `stream`.

- int **ppl_io_print_ConSys** (**ppl_const_ConSys_t** x)

Prints x *to* stdout.

- int **ppl_io_fprint_ConSys** (FILE *stream, **ppl_const_ConSys_t** x)

  Prints x *to the given output* stream*.*

- int **ppl_io_print_Generator** (**ppl_const_Generator_t** x)

  Prints x *to* stdout.

- int **ppl_io_fprint_Generator** (FILE *stream, **ppl_const_Generator_t** x)

  Prints x *to the given output* stream*.*

- int **ppl_io_print_GenSys** (**ppl_const_GenSys_t** x)

  Prints x *to* stdout.

- int **ppl_io_fprint_GenSys** (FILE *stream, **ppl_const_GenSys_t** x)

  Prints x *to the given output* stream*.*

- int **ppl_io_print_Polyhedron** (**ppl_const_Polyhedron_t** x)

  Prints x *to* stdout.

- int **ppl_io_fprint_Polyhedron** (FILE *stream, **ppl_const_Polyhedron_t** x)

  Prints x *to the given output* stream*.*

- int **ppl_io_set_variable_output_function** (**ppl_io_variable_output_function_type** *p)

  Sets the output function to be used for printing variables to p.

- int **ppl_io_get_variable_output_function** (**ppl_io_variable_output_function_type** **pp)

  Writes a pointer to the current variable output function to pp.

Initialization, Error Handling and Auxiliary Functions

- int **ppl_max_space_dimension** (**ppl_dimension_type** *m)

  Writes to m *the maximum space dimension this library can handle.*

- int **ppl_not_a_dimension** (**ppl_dimension_type** *m)

  Writes to m *a value that does not designate a valid dimension.*

- int **ppl_initialize** (void)

  Initializes the Parma Polyhedra Library. This function must be called before any other function.

- int **ppl_finalize** (void)

  Finalizes the Parma Polyhedra Library. This function must be called after any other function.

- int **ppl_set_error_handler** (void(*h)(enum **ppl_enum_error_code** code, const char *description))

  Installs the user-defined error handler pointed by h.

Functions Related to Coefficients

- int **ppl_new_Coefficient** (**ppl_Coefficient_t** ∗pc)

  Creates a new coefficient with value 0 and writes an handle for the newly created coefficient at address `pc`*.*

- int **ppl_new_Coefficient_from_mpz_t** (**ppl_Coefficient_t** ∗pc, mpz_t z)

  Creates a new coefficient with the value given by the GMP integer `z` *and writes an handle for the newly created coefficient at address* `pc`*.*

- int **ppl_new_Coefficient_from_Coefficient** (**ppl_Coefficient_t** ∗pc, **ppl_const_Coefficient_t** c)

  Builds a coefficient that is a copy of `c`*; writes an handle for the newly created coefficient at address* `pc`*.*

- int **ppl_assign_Coefficient_from_mpz_t** (**ppl_Coefficient_t** dst, mpz_t z)

  Assign to `dst` *the value given by the GMP integer* `z`*.*

- int **ppl_assign_Coefficient_from_Coefficient** (**ppl_Coefficient_t** dst, **ppl_const_Coefficient_t** src)

  Assigns a copy of the coefficient `src` *to* `dst`*.*

- int **ppl_delete_Coefficient** (**ppl_const_Coefficient_t** c)

  Invalidates the handle `c`*: this makes sure the corresponding resources will eventually be released.*

- int **ppl_Coefficient_to_mpz_t** (**ppl_const_Coefficient_t** c, mpz_t z)

  Sets the value of the GMP integer `z` *to the value of* `c`*.*

- int **ppl_Coefficient_OK** (**ppl_const_Coefficient_t** c)

  Returns a positive integer if `c` *is well formed, i.e., if it satisfies all its implementation invariants; returns 0 and perhaps make some noise if* `c` *is broken. Useful for debugging purposes.*

Functions Related to Linear Expressions

- int **ppl_new_LinExpression** (**ppl_LinExpression_t** ∗ple)

  Creates a new linear expression corresponding to the constant 0 in a zero-dimensional space; writes an handle for the new linear expression at address `ple`*.*

- int **ppl_new_LinExpression_with_dimension** (**ppl_LinExpression_t** ∗ple, **ppl_dimension_type** d)

  Creates a new linear expression corresponding to the constant 0 in a `d`*-dimensional space; writes an handle for the new linear expression at address* `ple`*.*

- int **ppl_new_LinExpression_from_LinExpression** (**ppl_LinExpression_t** ∗ple, **ppl_const_LinExpression_t** le)

  Builds a linear expression that is a copy of `le`*; writes an handle for the newly created linear expression at address* `ple`*.*

- int **ppl_new_LinExpression_from_Constraint** (**ppl_LinExpression_t** ∗ple, **ppl_const_Constraint_t** c)

  Builds a linear expression corresponding to constraint `c`*; writes an handle for the newly created linear expression at address* `ple`*.*

- int **ppl_new_LinExpression_from_Generator** (**ppl_LinExpression_t** *ple, **ppl_const_Generator_t** g)

  Builds a linear expression corresponding to generator `g`*; writes an handle for the newly created linear expression at address* `ple`.

- int **ppl_delete_LinExpression** (**ppl_const_LinExpression_t** le)

  Invalidates the handle `le`*: this makes sure the corresponding resources will eventually be released.*

- int **ppl_assign_LinExpression_from_LinExpression** (**ppl_LinExpression_t** dst, **ppl_const_LinExpression_t** src)

  Assigns a copy of the linear expression `src` *to* `dst`.

- int **ppl_LinExpression_add_to_coefficient** (**ppl_LinExpression_t** le, **ppl_dimension_type** var, **ppl_const_Coefficient_t** n)

  Adds `n` *to the coefficient of variable* `var` *in the linear expression* `le`*. The space dimension is set to be the maximum between* `var` *+ 1 and the old space dimension.*

- int **ppl_LinExpression_add_to_inhomogeneous** (**ppl_LinExpression_t** le, **ppl_const_Coefficient_t** n)

  Adds `n` *to the inhomogeneous term of the linear expression* `le`.

- int **ppl_add_LinExpression_to_LinExpression** (**ppl_LinExpression_t** dst, **ppl_const_LinExpression_t** src)

  Adds the linear expression `src` *to* `dst`.

- int **ppl_subtract_LinExpression_from_LinExpression** (**ppl_LinExpression_t** dst, **ppl_const_LinExpression_t** src)

  Subtracts the linear expression `src` *from* `dst`.

- int **ppl_multiply_LinExpression_by_Coefficient** (**ppl_LinExpression_t** le, **ppl_const_Coefficient_t** n)

  Multiply the linear expression `dst` *by* `n`.

- int **ppl_LinExpression_space_dimension** (**ppl_const_LinExpression_t** le)

  Returns the space dimension of `le`.

- int **ppl_LinExpression_coefficient** (**ppl_const_LinExpression_t** le, **ppl_dimension_type** var, **ppl_Coefficient_t** n)

  Copies into `n` *the coefficient of variable* `var` *in the linear expression* `le`.

- int **ppl_LinExpression_inhomogeneous_term** (**ppl_const_LinExpression_t** le, **ppl_Coefficient_t** n)

  Copies into `n` *the inhomogeneous term of linear expression* `le`.

- int **ppl_LinExpression_OK** (**ppl_const_LinExpression_t** le)

  Returns a positive integer if `le` *is well formed, i.e., if it satisfies all its implementation invariants; returns 0 and perhaps make some noise if* `le` *is broken. Useful for debugging purposes.*

Functions Related to Constraints

- int **ppl_new_Constraint** (**ppl_Constraint_t** $*$pc, **ppl_const_LinExpression_t** le, enum **ppl_-enum_Constraint_Type** rel)

  Creates the new constraint '`le rel 0`' and writes an handle for it at address `pc`*. The space dimension of the new constraint is equal to the space dimension of* `le`*.*

- int **ppl_new_Constraint_zero_dim_false** (**ppl_Constraint_t** $*$pc)

  Creates the unsatisfiable (zero-dimension space) constraint $0 = 1$ *and writes an handle for it at address* `pc`*.*

- int **ppl_new_Constraint_zero_dim_positivity** (**ppl_Constraint_t** $*$pc)

  Creates the true (zero-dimension space) constraint $0 \leq 1$*, also known as* positivity constraint*. An handle for the newly created constraint is written at address* `pc`*.*

- int **ppl_new_Constraint_from_Constraint** (**ppl_Constraint_t** $*$pc, **ppl_const_Constraint_t** c)

  Builds a constraint that is a copy of `c`*; writes an handle for the newly created constraint at address* `pc`*.*

- int **ppl_delete_Constraint** (**ppl_const_Constraint_t** c)

  Invalidates the handle `c:` *this makes sure the corresponding resources will eventually be released.*

- int **ppl_assign_Constraint_from_Constraint** (**ppl_Constraint_t** dst, **ppl_const_Constraint_t** src)

  Assigns a copy of the constraint `src` *to* `dst`*.*

- int **ppl_Constraint_space_dimension** (**ppl_const_Constraint_t** c)

  Returns the space dimension of `c`*.*

- int **ppl_Constraint_type** (**ppl_const_Constraint_t** c)

  Returns the type of constraint `c`*.*

- int **ppl_Constraint_coefficient** (**ppl_const_Constraint_t** c, **ppl_dimension_type** var, **ppl_-Coefficient_t** n)

  Copies into `n` *the coefficient of variable* `var` *in constraint* `c`*.*

- int **ppl_Constraint_inhomogeneous_term** (**ppl_const_Constraint_t** c, **ppl_Coefficient_t** n)

  Copies into `n` *the inhomogeneous term of constraint* `c`*.*

- int **ppl_Constraint_OK** (**ppl_const_Constraint_t** c)

  Returns a positive integer if `c` *is well formed, i.e., if it satisfies all its implementation invariants; returns 0 and perhaps make some noise if* `c` *is broken. Useful for debugging purposes.*

Functions Related to Constraint Systems

- int **ppl_new_ConSys** (**ppl_ConSys_t** $*$pcs)

  Builds an empty system of constraints and writes an handle to it at address `pcs`*.*

- int **ppl_new_ConSys_zero_dim_empty** (**ppl_ConSys_t** $*$pcs)

  Builds a zero-dimensional, unsatisfiable constraint system and writes an handle to it at address `pcs`*.*

- int **ppl_new_ConSys_from_Constraint** (**ppl_ConSys_t** *pcs, **ppl_const_Constraint_t** c)

  Builds the singleton constraint system containing only a copy of constraint `c`*; writes an handle for the newly created system at address* `pcs`.

- int **ppl_new_ConSys_from_ConSys** (**ppl_ConSys_t** *pcs, **ppl_const_ConSys_t** cs)

  Builds a constraint system that is a copy of `cs`*; writes an handle for the newly created system at address* `pcs`.

- int **ppl_delete_ConSys** (**ppl_const_ConSys_t** cs)

  Invalidates the handle `cs`*: this makes sure the corresponding resources will eventually be released.*

- int **ppl_assign_ConSys_from_ConSys** (**ppl_ConSys_t** dst, **ppl_const_ConSys_t** src)

  Assigns a copy of the constraint system `src` *to* `dst`.

- int **ppl_ConSys_space_dimension** (**ppl_const_ConSys_t** cs)

  Returns the dimension of the vector space enclosing `cs`.

- int **ppl_ConSys_clear** (**ppl_ConSys_t** cs)

  Removes all the constraints from the constraint system `cs` *and sets its space dimension to 0.*

- int **ppl_ConSys_insert_Constraint** (**ppl_ConSys_t** cs, **ppl_const_Constraint_t** c)

  Inserts a copy of the constraint `c` *into* `cs`*; the space dimension is increased, if necessary.*

- int **ppl_ConSys_OK** (**ppl_const_ConSys_t** c)

  Returns a positive integer if `cs` *is well formed, i.e., if it satisfies all its implementation invariants; returns 0 and perhaps make some noise if* `cs` *is broken. Useful for debugging purposes.*

- int **ppl_new_ConSys_const_iterator** (**ppl_ConSys_const_iterator_t** *pcit)

  Builds a new 'const iterator' and writes an handle to it at address `pcit`.

- int **ppl_new_ConSys_const_iterator_from_ConSys_const_iterator** (**ppl_ConSys_const_iterator_t** *pcit, **ppl_const_ConSys_const_iterator_t** cit)

  Builds a const iterator that is a copy of `cit`*; writes an handle for the newly created const iterator at address* `pcit`.

- int **ppl_delete_ConSys_const_iterator** (**ppl_const_ConSys_const_iterator_t** cit)

  Invalidates the handle `cit`*: this makes sure the corresponding resources will eventually be released.*

- int **ppl_assign_ConSys_const_iterator_from_ConSys_const_iterator** (**ppl_ConSys_const_iterator_t** dst, **ppl_const_ConSys_const_iterator_t** src)

  Assigns a copy of the const iterator `src` *to* `dst`.

- int **ppl_ConSys_begin** (**ppl_const_ConSys_t** cs, **ppl_ConSys_const_iterator_t** cit)

  Assigns to `cit` *a const iterator "pointing" to the beginning of the constraint system* `cs`.

- int **ppl_ConSys_end** (**ppl_const_ConSys_t** cs, **ppl_ConSys_const_iterator_t** cit)

  Assigns to `cit` *a const iterator "pointing" past the end of the constraint system* `cs`.

- int **ppl_ConSys_const_iterator_dereference** (**ppl_const_ConSys_const_iterator_t** cit, **ppl_const_Constraint_t** *pc)

  Dereference `cit` *writing a const handle to the resulting constraint at address* `pc`.

- int **ppl_ConSys_const_iterator_increment** (**ppl_ConSys_const_iterator_t** cit)

  Increment `cit` *so that it "points" to the next constraint.*

- int **ppl_ConSys_const_iterator_equal_test** (**ppl_const_ConSys_const_iterator_t** x, **ppl_const_-ConSys_const_iterator_t** y)

  Returns a positive integer if the iterators corresponding to `x` *and* `y` *are equal; return 0 if they are different.*

Functions Related to Generators

- int **ppl_new_Generator** (**ppl_Generator_t** $*$pg, **ppl_const_LinExpression_t** le, enum **ppl_-enum_Generator_Type** t, **ppl_const_Coefficient_t** d)

  Creates a new generator of direction `le` *and type* `t`*. If the generator to be created is a point or a closure point, the divisor* `d` *is applied to* `le`*. For other types of generators* `d` *is simply disregarded. An handle for the new generator is written at address* `pg`*. The space dimension of the new generator is equal to the space dimension of* `le`*.*

- int **ppl_new_Generator_zero_dim_point** (**ppl_Generator_t** $*$pg)

  Creates the point that is the origin of the zero-dimensional space $\mathbb{R}^0$*. Writes an handle for the new generator at address* `pg`*.*

- int **ppl_new_Generator_zero_dim_closure_point** (**ppl_Generator_t** $*$pg)

  Creates, as a closure point, the point that is the origin of the zero-dimensional space $\mathbb{R}^0$*. Writes an handle for the new generator at address* `pg`*.*

- int **ppl_new_Generator_from_Generator** (**ppl_Generator_t** $*$pg, **ppl_const_Generator_t** g)

  Builds a generator that is a copy of `g`*; writes an handle for the newly created generator at address* `pg`*.*

- int **ppl_delete_Generator** (**ppl_const_Generator_t** g)

  Invalidates the handle `g:` *this makes sure the corresponding resources will eventually be released.*

- int **ppl_assign_Generator_from_Generator** (**ppl_Generator_t** dst, **ppl_const_Generator_t** src)

  Assigns a copy of the generator `src` *to* `dst`*.*

- int **ppl_Generator_space_dimension** (**ppl_const_Generator_t** g)

  Returns the space dimension of `g`*.*

- int **ppl_Generator_type** (**ppl_const_Generator_t** g)

  Returns the type of generator `g`*.*

- int **ppl_Generator_coefficient** (**ppl_const_Generator_t** g, **ppl_dimension_type** var, **ppl_-Coefficient_t** n)

  Copies into `n` *the coefficient of variable* `var` *in generator* `g`*.*

- int **ppl_Generator_divisor** (**ppl_const_Generator_t** g, **ppl_Coefficient_t** n)

  If `g` *is a point or a closure point assigns its divisor to* `n`*.*

- int **ppl_Generator_OK** (**ppl_const_Generator_t** g)

  Returns a positive integer if `g` *is well formed, i.e., if it satisfies all its implementation invariants; returns 0 and perhaps make some noise if* `g` *is broken. Useful for debugging purposes.*

Functions Related to Generator Systems

- int **ppl_new_GenSys** (**ppl_GenSys_t** ∗pgs)

  Builds an empty system of generators and writes an handle to it at address `pgs`.

- int **ppl_new_GenSys_from_Generator** (**ppl_GenSys_t** ∗pgs, **ppl_const_Generator_t** g)

  Builds the singleton generator system containing only a copy of generator `g`*; writes an handle for the newly created system at address* `pgs`.

- int **ppl_new_GenSys_from_GenSys** (**ppl_GenSys_t** ∗pgs, **ppl_const_GenSys_t** gs)

  Builds a generator system that is a copy of `gs`*; writes an handle for the newly created system at address* `pgs`.

- int **ppl_delete_GenSys** (**ppl_const_GenSys_t** gs)

  Invalidates the handle `gs`*: this makes sure the corresponding resources will eventually be released.*

- int **ppl_assign_GenSys_from_GenSys** (**ppl_GenSys_t** dst, **ppl_const_GenSys_t** src)

  Assigns a copy of the generator system `src` *to* `dst`.

- int **ppl_GenSys_space_dimension** (**ppl_const_GenSys_t** gs)

  Returns the dimension of the vector space enclosing `gs`.

- int **ppl_GenSys_clear** (**ppl_GenSys_t** gs)

  Removes all the generators from the generator system `gs` *and sets its space dimension to 0.*

- int **ppl_GenSys_insert_Generator** (**ppl_GenSys_t** gs, **ppl_const_Generator_t** g)

  Inserts a copy of the generator `g` *into* `gs`*; the space dimension is increased, if necessary.*

- int **ppl_GenSys_OK** (**ppl_const_GenSys_t** c)

  Returns a positive integer if `gs` *is well formed, i.e., if it satisfies all its implementation invariants; returns 0 and perhaps make some noise if* `gs` *is broken. Useful for debugging purposes.*

- int **ppl_new_GenSys_const_iterator** (**ppl_GenSys_const_iterator_t** ∗pgit)

  Builds a new 'const iterator' and writes an handle to it at address `pgit`.

- int **ppl_new_GenSys_const_iterator_from_GenSys_const_iterator** (**ppl_GenSys_const_iterator_t** ∗pgit, **ppl_const_GenSys_const_iterator_t** git)

  Builds a const iterator that is a copy of `git`*; writes an handle for the newly created const iterator at address* `pgit`.

- int **ppl_delete_GenSys_const_iterator** (**ppl_const_GenSys_const_iterator_t** git)

  Invalidates the handle `git`*: this makes sure the corresponding resources will eventually be released.*

- int **ppl_assign_GenSys_const_iterator_from_GenSys_const_iterator** (**ppl_GenSys_const_iterator_t** dst, **ppl_const_GenSys_const_iterator_t** src)

  Assigns a copy of the const iterator `src` *to* `dst`.

- int **ppl_GenSys_begin** (**ppl_const_GenSys_t** gs, **ppl_GenSys_const_iterator_t** git)

  Assigns to `git` *a const iterator "pointing" to the beginning of the generator system* `gs`.

- int **ppl_GenSys_end** (**ppl_const_GenSys_t** gs, **ppl_GenSys_const_iterator_t** git)

Assigns to `git` *a const iterator "pointing" past the end of the generator system* `gs`*.*

- int **ppl_GenSys_const_iterator_dereference** (**ppl_const_GenSys_const_iterator_t** git, **ppl_const_Generator_t** ∗pg)

  Dereference `git` *writing a const handle to the resulting generator at address* `pg`*.*

- int **ppl_GenSys_const_iterator_increment** (**ppl_GenSys_const_iterator_t** git)

  Increment `git` *so that it "points" to the next generator.*

- int **ppl_GenSys_const_iterator_equal_test** (**ppl_const_GenSys_const_iterator_t** x, **ppl_const_GenSys_const_iterator_t** y)

  Return a positive integer if the iterators corresponding to `x` *and* `y` *are equal; return 0 if they are different.*

Functions Related to Polyhedra

- int **ppl_new_C_Polyhedron_from_dimension** (**ppl_Polyhedron_t** ∗pph, **ppl_dimension_type** d)

  Builds an universe closed polyhedron of dimension `d` *and writes an handle to it at address* `pph`*.*

- int **ppl_new_NNC_Polyhedron_from_dimension** (**ppl_Polyhedron_t** ∗pph, **ppl_dimension_type** d)

  Builds an universe NNC polyhedron of dimension `d` *and writes an handle to it at address* `pph`*.*

- int **ppl_new_C_Polyhedron_empty_from_dimension** (**ppl_Polyhedron_t** ∗pph, **ppl_dimension_type** d)

  Builds an empty closed polyhedron of dimension `d` *and writes an handle to it at address* `pph`*.*

- int **ppl_new_NNC_Polyhedron_empty_from_dimension** (**ppl_Polyhedron_t** ∗pph, **ppl_dimension_type** d)

  Builds an empty NNC polyhedron of dimension `d` *and writes an handle to it at address* `pph`*.*

- int **ppl_new_C_Polyhedron_from_C_Polyhedron** (**ppl_Polyhedron_t** ∗pph, **ppl_const_Polyhedron_t** ph)

  Builds a closed polyhedron that is a copy of `ph`*; writes an handle for the newly created polyhedron at address* `pph`*.*

- int **ppl_new_C_Polyhedron_from_NNC_Polyhedron** (**ppl_Polyhedron_t** ∗pph, **ppl_const_Polyhedron_t** ph)

  Builds a closed polyhedron that is a copy of of the NNC polyhedron `ph`*; writes an handle for the newly created polyhedron at address* `pph`*.*

- int **ppl_new_NNC_Polyhedron_from_C_Polyhedron** (**ppl_Polyhedron_t** ∗pph, **ppl_const_Polyhedron_t** ph)

  Builds an NNC polyhedron that is a copy of of the closed polyhedron `ph`*; writes an handle for the newly created polyhedron at address* `pph`*.*

- int **ppl_new_NNC_Polyhedron_from_NNC_Polyhedron** (**ppl_Polyhedron_t** ∗pph, **ppl_const_Polyhedron_t** ph)

  Builds an NNC polyhedron that is a copy of `ph`*; writes an handle for the newly created polyhedron at address* `pph`*.*

- int **ppl_new_C_Polyhedron_from_ConSys** (**ppl_Polyhedron_t** ∗pph, **ppl_const_ConSys_t** cs)

  Builds a new closed polyhedron from the system of constraints `cs` *and writes an handle for the newly created polyhedron at address* `pph`.

- int **ppl_new_C_Polyhedron_recycle_ConSys** (**ppl_Polyhedron_t** ∗pph, **ppl_ConSys_t** cs)

  Builds a new closed polyhedron recycling the system of constraints `cs` *and writes an handle for the newly created polyhedron at address* `pph`.

- int **ppl_new_NNC_Polyhedron_from_ConSys** (**ppl_Polyhedron_t** ∗pph, **ppl_const_ConSys_t** cs)

  Builds a new NNC polyhedron from the system of constraints `cs` *and writes an handle for the newly created polyhedron at address* `pph`.

- int **ppl_new_NNC_Polyhedron_recycle_ConSys** (**ppl_Polyhedron_t** ∗pph, **ppl_ConSys_t** cs)

  Builds a new NNC polyhedron recycling the system of constraints `cs` *and writes an handle for the newly created polyhedron at address* `pph`.

- int **ppl_new_C_Polyhedron_from_GenSys** (**ppl_Polyhedron_t** ∗pph, **ppl_const_GenSys_t** gs)

  Builds a new closed polyhedron from the system of generators `gs` *and writes an handle for the newly created polyhedron at address* `pph`.

- int **ppl_new_C_Polyhedron_recycle_GenSys** (**ppl_Polyhedron_t** ∗pph, **ppl_GenSys_t** gs)

  Builds a new closed polyhedron recycling the system of generators `gs` *and writes an handle for the newly created polyhedron at address* `pph`.

- int **ppl_new_NNC_Polyhedron_from_GenSys** (**ppl_Polyhedron_t** ∗pph, **ppl_const_GenSys_t** gs)

  Builds a new NNC polyhedron from the system of generators `gs` *and writes an handle for the newly created polyhedron at address* `pph`.

- int **ppl_new_NNC_Polyhedron_recycle_GenSys** (**ppl_Polyhedron_t** ∗pph, **ppl_GenSys_t** gs)

  Builds a new NNC polyhedron recycling the system of generators `gs` *and writes an handle for the newly created polyhedron at address* `pph`.

- int **ppl_new_C_Polyhedron_from_bounding_box** (**ppl_Polyhedron_t** ∗pph, **ppl_dimension_type**(∗space_dimension)(void), int(∗is_empty)(void), int(∗get_lower_bound)(**ppl_dimension_type** k, int closed, **ppl_Coefficient_t** n, **ppl_Coefficient_t** d), int(∗get_upper_bound)(**ppl_dimension_type** k, int closed, **ppl_Coefficient_t** n, **ppl_Coefficient_t** d))

  Builds a new C polyhedron corresponding to an interval-based bounding box, writing a handle for the newly created polyhedron at address `pph`.

- int **ppl_new_NNC_Polyhedron_from_bounding_box** (**ppl_Polyhedron_t** ∗pph, **ppl_dimension_type**(∗space_dimension)(void), int(∗is_empty)(void), int(∗get_lower_bound)(**ppl_dimension_type** k, int closed, **ppl_Coefficient_t** n, **ppl_Coefficient_t** d), int(∗get_upper_bound)(**ppl_dimension_type** k, int closed, **ppl_Coefficient_t** n, **ppl_Coefficient_t** d))

  Builds a new C polyhedron corresponding to an interval-based bounding box, writing a handle for the newly created polyhedron at address `pph`.

- int **ppl_assign_C_Polyhedron_from_C_Polyhedron** (**ppl_Polyhedron_t** dst, **ppl_const_Polyhedron_t** src)

  Assigns a copy of the closed polyhedron `src` *to the closed polyhedron* `dst`.

- int **ppl_assign_NNC_Polyhedron_from_NNC_Polyhedron** (**ppl_Polyhedron_t** dst, **ppl_const_-Polyhedron_t** src)

  Assigns a copy of the NNC polyhedron `src` *to the NNC polyhedron* `dst`.

- int **ppl_delete_Polyhedron** (**ppl_const_Polyhedron_t** ph)

  Invalidates the handle `ph`*: this makes sure the corresponding resources will eventually be released.*

- int **ppl_Polyhedron_space_dimension** (**ppl_const_Polyhedron_t** ph)

  Returns the dimension of the vector space enclosing `ph`.

- int **ppl_Polyhedron_constraints** (**ppl_const_Polyhedron_t** ph, **ppl_const_ConSys_t** ∗pcs)

  Writes a const handle to the constraint system defining the polyhedron `ph` *at address* `pcs`.

- int **ppl_Polyhedron_minimized_constraints** (**ppl_const_Polyhedron_t** ph, **ppl_const_ConSys_t** ∗pcs)

  Writes a const handle to the minimized constraint system defining the polyhedron `ph` *at address* `pcs`.

- int **ppl_Polyhedron_generators** (**ppl_const_Polyhedron_t** ph, **ppl_const_GenSys_t** ∗pgs)

  Writes a const handle to the generator system defining the polyhedron `ph` *at address* `pgs`.

- int **ppl_Polyhedron_minimized_generators** (**ppl_const_Polyhedron_t** ph, **ppl_const_GenSys_t** ∗pgs)

  Writes a const handle to the minimized generator system defining the polyhedron `ph` *at address* `pgs`.

- int **ppl_Polyhedron_relation_with_Constraint** (**ppl_const_Polyhedron_t** ph, **ppl_const_-Constraint_t** c)

  Checks the relation between the polyhedron `ph` *with the constraint* `c`.

- int **ppl_Polyhedron_relation_with_Generator** (**ppl_const_Polyhedron_t** ph, **ppl_const_-Generator_t** g)

  Checks the relation between the polyhedron `ph` *with the generator* `g`.

- int **ppl_Polyhedron_shrink_bounding_box** (**ppl_const_Polyhedron_t** ph, unsigned int complexity, void(∗set_empty)(void), void(∗raise_lower_bound)(**ppl_dimension_type** k, int closed, **ppl_-const_Coefficient_t** n, **ppl_const_Coefficient_t** d), void(∗lower_upper_bound)(**ppl_dimension_-type** k, int closed, **ppl_const_Coefficient_t** n, **ppl_const_Coefficient_t** d))

  Use `ph` *to shrink a generic, interval-based bounding box. The bounding box is abstractly provided by means of the parameters.*

- int **ppl_Polyhedron_is_empty** (**ppl_const_Polyhedron_t** ph)

  Returns a positive integer if `ph` *is empty; returns 0 if* `ph` *is not empty.*

- int **ppl_Polyhedron_is_universe** (**ppl_const_Polyhedron_t** ph)

  Returns a positive integer if `ph` *is a universe polyhedron; returns 0 if it is not.*

- int **ppl_Polyhedron_is_bounded** (**ppl_const_Polyhedron_t** ph)

  Returns a positive integer if `ph` *is bounded; returns 0 if* `ph` *is unbounded.*

- int **ppl_Polyhedron_bounds_from_above** (**ppl_const_Polyhedron_t** ph, **ppl_const_Lin-Expression_t** le)

  Returns a positive integer if `le` *is bounded from above in* `ph`*; returns 0 otherwise.*

- int **ppl_Polyhedron_bounds_from_below** (**ppl_const_Polyhedron_t** ph, **ppl_const_LinExpression_t** le)

  Returns a positive integer if `le` *is bounded from below in* `ph`*; returns 0 otherwise.*

- int **ppl_Polyhedron_maximize** (**ppl_const_Polyhedron_t** ph, **ppl_const_LinExpression_t** le, **ppl_Coefficient_t** sup_n, **ppl_Coefficient_t** sup_d, int ∗pmaximum, **ppl_const_Generator_t** ∗ppoint)

  Returns a positive integer if `ph` *is not empty and* `le` *is bounded from above in* `ph`*, in which case the supremum value and a point where* `le` *reaches it are computed.*

- int **ppl_Polyhedron_minimize** (**ppl_const_Polyhedron_t** ph, **ppl_const_LinExpression_t** le, **ppl_Coefficient_t** inf_n, **ppl_Coefficient_t** inf_d, int ∗pminimum, **ppl_const_Generator_t** ∗ppoint)

  Returns a positive integer if `ph` *is not empty and* `le` *is bounded from above in* `ph`*, in which case the infimum value and a point where* `le` *reaches it are computed.*

- int **ppl_Polyhedron_is_topologically_closed** (**ppl_const_Polyhedron_t** ph)

  Returns a positive integer if `ph` *is topologically closed; returns 0 if* `ph` *is not topologically closed.*

- int **ppl_Polyhedron_contains_Polyhedron** (**ppl_const_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  Returns a positive integer if `x` *contains or is equal to* `y`*; returns 0 if it does not.*

- int **ppl_Polyhedron_strictly_contains_Polyhedron** (**ppl_const_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  Returns a positive integer if `x` *strictly contains* `y`*; returns 0 if it does not.*

- int **ppl_Polyhedron_is_disjoint_from_Polyhedron** (**ppl_const_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  Returns a positive integer if `x` *and* `y` *are disjoint; returns 0 if they are not.*

- int **ppl_Polyhedron_equals_Polyhedron** (**ppl_const_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  Returns a positive integer if `x` *and* `y` *are the same polyhedron; return 0 if they are different.*

- int **ppl_Polyhedron_OK** (**ppl_const_Polyhedron_t** ph)

  Returns a positive integer if `ph` *is well formed, i.e., if it satisfies all its implementation invariants; returns 0 and perhaps make some noise if* `ph` *is broken. Useful for debugging purposes.*

- int **ppl_Polyhedron_add_constraint** (**ppl_Polyhedron_t** ph, **ppl_const_Constraint_t** c)

  Adds a copy of the constraint `c` *to the system of constraints of* `ph`*.*

- int **ppl_Polyhedron_add_constraint_and_minimize** (**ppl_Polyhedron_t** ph, **ppl_const_Constraint_t** c)

  Adds a copy of the constraint `c` *to the system of constraints of* `ph`*. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return,* `ph` *is guaranteed to be minimized.*

- int **ppl_Polyhedron_add_generator** (**ppl_Polyhedron_t** ph, **ppl_const_Generator_t** g)

  Adds a copy of the generator `g` *to the system of generators of* `ph`*.*

- int **ppl_Polyhedron_add_generator_and_minimize** (**ppl_Polyhedron_t** ph, **ppl_const_-Generator_t** g)

  Adds a copy of the generator g *to the system of generators of* ph*. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return,* ph *is guaranteed to be minimized.*

- int **ppl_Polyhedron_add_constraints** (**ppl_Polyhedron_t** ph, **ppl_const_ConSys_t** cs)

  Adds a copy of the system of constraints cs *to the system of constraints of* ph*.*

- int **ppl_Polyhedron_add_constraints_and_minimize** (**ppl_Polyhedron_t** ph, **ppl_const_Con-Sys_t** cs)

  Adds a copy of the system of constraints cs *to the system of constraints of* ph*. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return,* ph *is guaranteed to be minimized.*

- int **ppl_Polyhedron_add_generators** (**ppl_Polyhedron_t** ph, **ppl_const_GenSys_t** gs)

  Adds a copy of the system of generators gs *to the system of generators of* ph*.*

- int **ppl_Polyhedron_add_generators_and_minimize** (**ppl_Polyhedron_t** ph, **ppl_const_Gen-Sys_t** gs)

  Adds a copy of the system of generators gs *to the system of generators of* ph*. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return,* ph *is guaranteed to be minimized.*

- int **ppl_Polyhedron_add_recycled_constraints** (**ppl_Polyhedron_t** ph, **ppl_ConSys_t** cs)

  Adds the system of constraints cs *to the system of constraints of* ph*.*

- int **ppl_Polyhedron_add_recycled_constraints_and_minimize** (**ppl_Polyhedron_t** ph, **ppl_Con-Sys_t** cs)

  Adds the system of constraints cs *to the system of constraints of* ph*. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return,* ph *is guaranteed to be minimized.*

- int **ppl_Polyhedron_add_recycled_generators** (**ppl_Polyhedron_t** ph, **ppl_GenSys_t** gs)

  Adds the system of generators gs *to the system of generators of* ph*.*

- int **ppl_Polyhedron_add_recycled_generators_and_minimize** (**ppl_Polyhedron_t** ph, **ppl_Gen-Sys_t** gs)

  Adds the system of generators gs *to the system of generators of* ph*. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return,* ph *is guaranteed to be minimized.*

- int **ppl_Polyhedron_intersection_assign** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  Intersects x *with polyhedron* y *and assigns the result* x*.*

- int **ppl_Polyhedron_intersection_assign_and_minimize** (**ppl_Polyhedron_t** x, **ppl_const_-Polyhedron_t** y)

  Intersects x *with polyhedron* y *and assigns the result* x*. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return,* x *is also guaranteed to be minimized.*

- int **ppl_Polyhedron_poly_hull_assign** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  Assigns to x *the poly-hull of* x *and* y*.*

- int **ppl_Polyhedron_poly_hull_assign_and_minimize** (**ppl_Polyhedron_t** x, **ppl_const_-Polyhedron_t** y)

  Assigns to `x` *the poly-hull of* `x` *and* `y`*. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return,* `x` *is also guaranteed to be minimized.*

- int **ppl_Polyhedron_poly_difference_assign** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  Assigns to `x` *the* **poly-difference**(p. 6) *of* `x` *and* `y`*.*

- int **ppl_Polyhedron_affine_image** (**ppl_Polyhedron_t** ph, **ppl_dimension_type** var, **ppl_const_-LinExpression_t** le, **ppl_const_Coefficient_t** d)

  Transforms the polyhedron `ph`*, assigning an affine expression to the specified variable.*

- int **ppl_Polyhedron_affine_preimage** (**ppl_Polyhedron_t** ph, **ppl_dimension_type** var, **ppl_-const_LinExpression_t** le, **ppl_const_Coefficient_t** d)

  Transforms the polyhedron `ph`*, substituting an affine expression to the specified variable.*

- int **ppl_Polyhedron_generalized_affine_image** (**ppl_Polyhedron_t** ph, **ppl_dimension_type** var, enum **ppl_enum_Constraint_Type** relsym, **ppl_const_LinExpression_t** le, **ppl_const_-Coefficient_t** d)

  Assigns to `ph` *the image of* `ph` *with respect to the* **generalized affine transfer function**(p. 9) $\mathrm{var}' \bowtie \frac{\mathrm{expr}}{\mathrm{denominator}}$*, where* $\bowtie$ *is the relation symbol encoded by* `relsym`*.*

- int **ppl_Polyhedron_generalized_affine_image_lhs_rhs** (**ppl_Polyhedron_t** ph, **ppl_const_Lin-Expression_t** lhs, enum **ppl_enum_Constraint_Type** relsym, **ppl_const_LinExpression_t** rhs)

  Assigns to `ph` *the image of* `ph` *with respect to the* **generalized affine transfer function**(p. 9) $\mathrm{lhs}' \bowtie \mathrm{rhs}$*, where* $\bowtie$ *is the relation symbol encoded by* `relsym`*.*

- int **ppl_Polyhedron_time_elapse_assign** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  Assigns to `x` *the* **time-elapse**(p. 9) *between the polyhedra* `x` *and* `y`*.*

- int **ppl_Polyhedron_BHRZ03_widening_assign_with_tokens** (**ppl_Polyhedron_t** x, **ppl_const_-Polyhedron_t** y, unsigned ∗tp)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **BHRZ03-widening**(p. 10) *of* `x` *and* `y`*. If* `tp` *is not the null pointer, the* **widening with tokens**(p. 11) *delay technique is applied with* `*tp` *available tokens.*

- int **ppl_Polyhedron_BHRZ03_widening_assign** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **BHRZ03-widening**(p. 10) *of* `x` *and* `y`*.*

- int **ppl_Polyhedron_limited_BHRZ03_extrapolation_assign_with_tokens** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y, **ppl_const_ConSys_t** cs, unsigned ∗tp)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **BHRZ03-widening**(p. 10) *of* `x` *and* `y` *intersected with the constraints in* `cs` *that are satisfied by all the points of* `x`*. If* `tp` *is not the null pointer, the* **widening with tokens**(p. 11) *delay technique is applied with* `*tp` *available tokens.*

- int **ppl_Polyhedron_limited_BHRZ03_extrapolation_assign** (**ppl_Polyhedron_t** x, **ppl_const_-Polyhedron_t** y, **ppl_const_ConSys_t** cs)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **BHRZ03-widening**(p. 10) *of* `x` *and* `y` *intersected with the constraints in* `cs` *that are satisfied by all the points of* `x`*.*

- int **ppl_Polyhedron_bounded_BHRZ03_extrapolation_assign_with_tokens** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y, **ppl_const_ConSys_t** cs, unsigned ∗tp)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **BHRZ03-widening**(p. 10) *of* `x` *and* `y` *intersected with the constraints in* `cs` *that are satisfied by all the points of* `x`*, further intersected with all the constraints of the form* $\pm v \leq r$ *and* $\pm v < r$*, with* $r \in \mathbb{Q}$*, that are satisfied by all the points of* `x`*. If* `tp` *is not the null pointer, the* **widening with tokens**(p. 11) *delay technique is applied with* `*tp` *available tokens.*

- int **ppl_Polyhedron_bounded_BHRZ03_extrapolation_assign** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y, **ppl_const_ConSys_t** cs)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **BHRZ03-widening**(p. 10) *of* `x` *and* `y` *intersected with the constraints in* `cs` *that are satisfied by all the points of* `x`*, further intersected with all the constraints of the form* $\pm v \leq r$ *and* $\pm v < r$*, with* $r \in \mathbb{Q}$*, that are satisfied by all the points of* `x`*.*

- int **ppl_Polyhedron_H79_widening_assign_with_tokens** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y, unsigned ∗tp)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **H79-widening**(p. 10) *of* `x` *and* `y`*. If* `tp` *is not the null pointer, the* **widening with tokens**(p. 11) *delay technique is applied with* `*tp` *available tokens.*

- int **ppl_Polyhedron_H79_widening_assign** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **H79-widening**(p. 10) *of* `x` *and* `y`*.*

- int **ppl_Polyhedron_limited_H79_extrapolation_assign_with_tokens** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y, **ppl_const_ConSys_t** cs, unsigned ∗tp)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **H79-widening**(p. 10) *of* `x` *and* `y` *intersected with the constraints in* `cs` *that are satisfied by all the points of* `x`*. If* `tp` *is not the null pointer, the* **widening with tokens**(p. 11) *delay technique is applied with* `*tp` *available tokens.*

- int **ppl_Polyhedron_limited_H79_extrapolation_assign** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y, **ppl_const_ConSys_t** cs)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **H79-widening**(p. 10) *of* `x` *and* `y` *intersected with the constraints in* `cs` *that are satisfied by all the points of* `x`*.*

- int **ppl_Polyhedron_bounded_H79_extrapolation_assign_with_tokens** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y, **ppl_const_ConSys_t** cs, unsigned ∗tp)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **H79-widening**(p. 10) *of* `x` *and* `y` *intersected with the constraints in* `cs` *that are satisfied by all the points of* `x`*, further intersected with all the constraints of the form* $\pm v \leq r$ *and* $\pm v < r$*, with* $r \in \mathbb{Q}$*, that are satisfied by all the points of* `x`*. If* `tp` *is not the null pointer, the* **widening with tokens**(p. 11) *delay technique is applied with* `*tp` *available tokens.*

- int **ppl_Polyhedron_bounded_H79_extrapolation_assign** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y, **ppl_const_ConSys_t** cs)

  If the polyhedron `y` *is contained in (or equal to) the polyhedron* `x`*, assigns to* `x` *the* **H79-widening**(p. 10) *of* `x` *and* `y` *intersected with the constraints in* `cs` *that are satisfied by all the points of* `x`*, further intersected with all the constraints of the form* $\pm v \leq r$ *and* $\pm v < r$*, with* $r \in \mathbb{Q}$*, that are satisfied by all the points of* `x`*.*

- int **ppl_Polyhedron_topological_closure_assign** (**ppl_Polyhedron_t** ph)

Assigns to `ph` *its topological closure.*

- int **ppl_Polyhedron_add_dimensions_and_embed** (**ppl_Polyhedron_t** ph, **ppl_dimension_type** d)

  Adds `d` *new dimensions to the space enclosing the polyhedron* `ph` *and to* `ph` *itself.*

- int **ppl_Polyhedron_add_dimensions_and_project** (**ppl_Polyhedron_t** ph, **ppl_dimension_type** d)

  Adds `d` *new dimensions to the space enclosing the polyhedron* `ph`.

- int **ppl_Polyhedron_concatenate_assign** (**ppl_Polyhedron_t** x, **ppl_const_Polyhedron_t** y)

  Seeing a polyhedron as a set of tuples (its points), assigns to `x` *all the tuples that can be obtained by concatenating, in the order given, a tuple of* `x` *with a tuple of* `y`.

- int **ppl_Polyhedron_remove_dimensions** (**ppl_Polyhedron_t** ph, **ppl_dimension_type** ds[ ], size_t n)

  Removes from `ph` *and its containing space the dimensions that are specified in first* `n` *positions of the array* `ds`. *The presence of duplicates in* `ds` *is a waste but an innocuous one.*

- int **ppl_Polyhedron_remove_higher_dimensions** (**ppl_Polyhedron_t** ph, **ppl_dimension_type** d)

  Removes the higher dimensions from `ph` *and its enclosing space so that, upon successful return, the new space dimension is* `d`.

- int **ppl_Polyhedron_map_dimensions** (**ppl_Polyhedron_t** ph, **ppl_dimension_type** maps[ ], size_t n)

  Remaps the dimensions of the vector space according to a **partial function**(p. 7). *This function is specified by means of the* `maps` *array, which has* `n` *entries.*

- int **ppl_Polyhedron_expand_dimension** (**ppl_Polyhedron_t** ph, **ppl_dimension_type** d, **ppl_dimension_type** m)

  Expands(p. 7) *the* `d`*-th dimension of* `ph` *to* `m` *new dimensions.*

- int **ppl_Polyhedron_fold_dimensions** (**ppl_Polyhedron_t** ph, **ppl_dimension_type** ds[ ], size_t n, **ppl_dimension_type** d)

  Modifies `ph` *by* **folding**(p. 8) *the dimensions contained in the first* `n` *positions of the array* `ds` *into dimension* `d`. *The presence of duplicates in* `ds` *is a waste but an innocuous one.*

Typedefs

- typedef size_t **ppl_dimension_type**

  An unsigned integral type for representing space dimensions.

- typedef ppl_Coefficient_tag ∗ **ppl_Coefficient_t**

  Opaque pointer.

- typedef ppl_Coefficient_tag const ∗ **ppl_const_Coefficient_t**

  Opaque pointer to const object.

- typedef ppl_LinExpression_tag ∗ **ppl_LinExpression_t**

  Opaque pointer.

- typedef ppl_LinExpression_tag const * **ppl_const_LinExpression_t**

    Opaque pointer to const object.

- typedef ppl_Constraint_tag * **ppl_Constraint_t**

    Opaque pointer.

- typedef ppl_Constraint_tag const * **ppl_const_Constraint_t**

    Opaque pointer to const object.

- typedef ppl_ConSys_tag * **ppl_ConSys_t**

    Opaque pointer.

- typedef ppl_ConSys_tag const * **ppl_const_ConSys_t**

    Opaque pointer to const object.

- typedef ppl_ConSys_const_iterator_tag * **ppl_ConSys_const_iterator_t**

    Opaque pointer.

- typedef ppl_ConSys_const_iterator_tag const * **ppl_const_ConSys_const_iterator_t**

    Opaque pointer to const object.

- typedef ppl_Generator_tag * **ppl_Generator_t**

    Opaque pointer.

- typedef ppl_Generator_tag const * **ppl_const_Generator_t**

    Opaque pointer to const object.

- typedef ppl_GenSys_tag * **ppl_GenSys_t**

    Opaque pointer.

- typedef ppl_GenSys_tag const * **ppl_const_GenSys_t**

    Opaque pointer to const object.

- typedef ppl_GenSys_const_iterator_tag * **ppl_GenSys_const_iterator_t**

    Opaque pointer.

- typedef ppl_GenSys_const_iterator_tag const * **ppl_const_GenSys_const_iterator_t**

    Opaque pointer to const object.

- typedef ppl_Polyhedron_tag * **ppl_Polyhedron_t**

    Opaque pointer.

- typedef ppl_Polyhedron_tag const * **ppl_const_Polyhedron_t**

    Opaque pointer to const object.

Enumerations

- enum **ppl_enum_error_code** {
  PPL_ERROR_OUT_OF_MEMORY, **PPL_ERROR_INVALID_ARGUMENT**, **PPL_ERROR_LENGTH_ERROR**, **PPL_ARITHMETIC_OVERFLOW**,
  PPL_STDIO_ERROR, **PPL_ERROR_INTERNAL_ERROR**, **PPL_ERROR_UNKNOWN_STANDARD_EXCEPTION**, **PPL_ERROR_UNEXPECTED_ERROR** }

  Defines the error codes that any function may return.

- enum **ppl_enum_Constraint_Type** {
  PPL_CONSTRAINT_TYPE_LESS_THAN, **PPL_CONSTRAINT_TYPE_LESS_THAN_OR_EQUAL**, **PPL_CONSTRAINT_TYPE_EQUAL**, **PPL_CONSTRAINT_TYPE_GREATER_THAN_OR_EQUAL**,
  PPL_CONSTRAINT_TYPE_GREATER_THAN }

  Describes the relations represented by a constraint.

- enum **ppl_enum_Generator_Type** { **PPL_GENERATOR_TYPE_LINE**, **PPL_GENERATOR_TYPE_RAY**, **PPL_GENERATOR_TYPE_POINT**, **PPL_GENERATOR_TYPE_CLOSURE_POINT** }

  Describes the different kinds of generators.

Variables

- unsigned int **PPL_COMPLEXITY_CLASS_POLYNOMIAL**

  Code of the worst-case polynomial complexity class.

- unsigned int **PPL_COMPLEXITY_CLASS_SIMPLEX**

  Code of the worst-case exponential but typically polynomial complexity class.

- unsigned int **PPL_COMPLEXITY_CLASS_ANY**

  Code of the universal complexity class.

- unsigned int **PPL_POLY_CON_RELATION_IS_DISJOINT**

  Individual bit saying that the polyhedron and the set of points satisfying the constraint are disjoint.

- unsigned int **PPL_POLY_CON_RELATION_STRICTLY_INTERSECTS**

  Individual bit saying that the polyhedron intersects the set of points satisfying the constraint, but it is not included in it.

- unsigned int **PPL_POLY_CON_RELATION_IS_INCLUDED**

  Individual bit saying that the polyhedron is included in the set of points satisfying the constraint.

- unsigned int **PPL_POLY_CON_RELATION_SATURATES**

  Individual bit saying that the polyhedron is included in the set of points saturating the constraint.

- unsigned int **PPL_POLY_GEN_RELATION_SUBSUMES**

  Individual bit saying that adding the generator would not change the polyhedron.

7.3.1 Detailed Description

Some details about the C Interface(p. 40).

All the declarations needed for using the PPL's C interface (preprocessor symbols, data types, variables and functions) are collected in the header file **ppl_c.h**(p. **??**). This file, which is designed to work with pre-ANSI and ANSI C compilers as well as C99 and C++ compilers, should be included, either directly or via some other header file, with the directive

```
#include <ppl_c.h>
```

If this directive does not work, then your compiler is unable to find the file **ppl_c.h**(p. **??**). So check that the library is installed (if it is not installed, you may want to `make install`, perhaps with root privileges); that it is installed in the right place (if not you may want to reconfigure the library using the appropriate pathname for the `-prefix` option); and that your compiler knows where it is installed (if not you should add the path to the directory where **ppl_c.h**(p. **??**) is located to the compiler's include file search path; this is usually done with the `-I` option).

The name space of the PPL's C interface is `PPL_*` for preprocessor symbols, enumeration values and variables; and `ppl_*` for data types and function names. The interface systematically uses *opaque data types* (generic pointers that completely hide the internal representations from the client code) and provides all required access functions. By using just the interface, the client code can exploit all the functionalities of the library yet avoid directly manipulating the library's data structures. The advantages are that (1) applications do not depend on the internals of the library (these may change from release to release), and (2) the interface invariants can be thoroughly checked (by the access functions).

The PPL's C interface is initialized by means of the `ppl_initialize` function. This function must be called *before using any other interface of the library*. The application can release the resources allocated by the library by calling the `ppl_finalize` function. After this function is called *no other interface of the library may be used* until the interface is re-initialized using `ppl_initialize`.

Any application using the PPL should make sure that only the intended version(s) of the library are ever used. The version used can be checked at compile-time thanks to the macros PPL_VERSION_MAJOR, PPL_VERSION_MINOR, PPL_VERSION_REVISION and PPL_VERSION_BETA, which give, respectively major, minor, revision and beta numbers of the PPL version. This is an example of their use:

```
#if PPL_VERSION_MAJOR == 0 && PPL_VERSION_MINOR < 6
# error "PPL version 0.6 or following is required"
#endif
```

Compile-time checking, however, is not normally enough, particularly in an environment where there is dynamic linking. Run-time checking can be performed by means of the functions `ppl_version_major`, `ppl_version_minor`, `ppl_version_revision`, and `ppl_version_beta`. The PPL's C interface also provides functions `ppl_version`, returning character string containing the full version number, and `ppl_banner`, returning a string that, in addition, provides (pointers to) other useful information for the library user.

All programs using the PPL's C interface must link with the following libraries: `libppl_c` (PPL's C interface), `libppl` (PPL's core), `libgmpxx` (GMP's C++ interface), and `libgmp` (GMP's library core). On most Unix-like systems, this is done by adding `-lppl_c`, `-lppl`, `-lgmpxx`, and `-lgmp` to the compiler's or linker's command line. For example:

```
gcc myprogram.o -lppl_c -lppl -lgmpxx -lgmp
```

If this does not work, it means that your compiler/linker is not finding the libraries where it expects. Again, this could be because you forgot to install the library or you installed it in a non-standard location. In the latter case you will need to use the appropriate options (usually `-L`) and, if you use shared libraries, some sort of run-time path selection mechanisms. Consult your compiler's documentation for details. Notice that the PPL is built using `Libtool` and an application can exploit this fact to significantly simplify the linking phase. See Libtool's documentation for details. Those working under Linux can find a lot of useful information on how to use program libraries (including static, shared, and dynamically loaded libraries) in the `Program Library HOWTO`.

For examples on how to use the functions provided by the C interface, you are referred to the `interfaces/C/lpenum/` directory in the source distribution. It contains a toy *Linear Programming* solver written in C. In order to use this solver you will need to install `GLPK` (the GNU Linear Programming Kit): this is used to read linear programs in MPS format.

7.3.2 Define Documentation

7.3.2.1 #define PPL_VERSION "0.6.1"

A string containing the PPL version.

Let `M` and `m` denote the numbers associated to PPL_VERSION_MAJOR and PPL_VERSION_MINOR, respectively. The format of PPL_VERSION is `M "." m` if both PPL_VERSION_REVISION (`r`) and PPL_VERSION_BETA (`b`)are zero, `M "." m "pre" b` if PPL_VERSION_REVISION is zero and PPL_VERSION_BETA is not zero, `M "." m "." r` if PPL_VERSION_REVISION is not zero and PPL_VERSION_BETA is zero, `M "." m "." r "pre" b` if neither PPL_VERSION_REVISION nor PPL_VERSION_BETA are zero.

7.3.3 Typedef Documentation

7.3.3.1 typedef const char∗ ppl_io_variable_output_function_type(ppl_dimension_type var)

The type of output functions used for printing variables.

An output function for variables must write a textual representation for `var` to a character buffer, null-terminate it, and return a pointer to the beginning of the buffer. In case the operation fails, 0 should be returned and perhaps `errno` should be set in a meaningful way. The library does nothing with the buffer, besides printing its contents.

7.3.4 Enumeration Type Documentation

7.3.4.1 enum ppl_enum_error_code

Defines the error codes that any function may return.

Enumeration values:

PPL_ERROR_OUT_OF_MEMORY The virtual memory available to the process has been exhausted.

PPL_ERROR_INVALID_ARGUMENT A function has been invoked with an invalid argument.

PPL_ERROR_LENGTH_ERROR The construction of an object that would exceed its maximum permitted size was attempted.

PPL_ARITHMETIC_OVERFLOW An arithmetic overflow occurred and the computation was consequently interrupted. This can *only* happen in library's incarnations using bounded integers as coefficients.

PPL_STDIO_ERROR An error occurred during a C input/output operation. A more precise indication of what went wrong is availble via `errno`.

PPL_ERROR_INTERNAL_ERROR An internal error that was diagnosed by the PPL itself. This indicates a bug in the PPL.

PPL_ERROR_UNKNOWN_STANDARD_EXCEPTION A standard exception has been raised by the C++ run-time environment. This indicates a bug in the PPL.

PPL_ERROR_UNEXPECTED_ERROR A totally unknown, totally unexpected error happened. This indicates a bug in the PPL.

7.3.4.2 enum ppl_enum_Constraint_Type

Describes the relations represented by a constraint.

Enumeration values:

PPL_CONSTRAINT_TYPE_LESS_THAN The constraint is of the form $e < 0$.

PPL_CONSTRAINT_TYPE_LESS_THAN_OR_EQUAL The constraint is of the form $e \leq 0$.

PPL_CONSTRAINT_TYPE_EQUAL The constraint is of the form $e = 0$.

PPL_CONSTRAINT_TYPE_GREATER_THAN_OR_EQUAL The constraint is of the form $e \geq 0$.

PPL_CONSTRAINT_TYPE_GREATER_THAN The constraint is of the form $e > 0$.

7.3.4.3 enum ppl_enum_Generator_Type

Describes the different kinds of generators.

Enumeration values:

PPL_GENERATOR_TYPE_LINE The generator is a line.

PPL_GENERATOR_TYPE_RAY The generator is a ray.

PPL_GENERATOR_TYPE_POINT The generator is a point.

PPL_GENERATOR_TYPE_CLOSURE_POINT The generator is a closure point.

7.3.5 Function Documentation

7.3.5.1 int ppl_banner (const char ** *p*)

Writes to `m` a pointer to a character string containing the PPL banner.

The banner provides information about the PPL version, the licensing, the lack of any warranty whatsoever, the C++ compiler used to build the library, where to report bugs and where to look for further information.

7.3.5.2 int ppl_initialize (void)

Initializes the Parma Polyhedra Library. This function must be called before any other function.

Returns:

`PPL_ERROR_INVALID_ARGUMENT` if the library was already initialized.

7.3.5.3 int ppl_finalize (void)

Finalizes the Parma Polyhedra Library. This function must be called after any other function.

Returns:

`PPL_ERROR_INVALID_ARGUMENT` if the library was already finalized.

7.3.5.4 int ppl_set_error_handler (void(∗ *h*)(enum ppl_enum_error_code code, const char ∗description))

Installs the user-defined error handler pointed by `h`.

The error handler takes an error code and a textual description that gives further information about the actual error. The C string containing the textual description is read-only and its existence it not guaranteed after the handler has returned.

7.3.5.5 int ppl_new_C_Polyhedron_from_ConSys (ppl_Polyhedron_t ∗ *pph*, ppl_const_ConSys_t *cs*)

Builds a new closed polyhedron from the system of constraints `cs` and writes an handle for the newly created polyhedron at address `pph`.

The new polyhedron will inherit the space dimension of `cs`.

7.3.5.6 int ppl_new_C_Polyhedron_recycle_ConSys (ppl_Polyhedron_t ∗ *pph*, ppl_ConSys_t *cs*)

Builds a new closed polyhedron recycling the system of constraints `cs` and writes an handle for the newly created polyhedron at address `pph`.

Since `cs` will be *the* system of constraints of the new polyhedron, the space dimension is also inherited.

Warning:

This function modifies the constraint system referenced by `cs:` upon return, no assumption can be made on its value.

7.3.5.7 int ppl_new_NNC_Polyhedron_from_ConSys (ppl_Polyhedron_t ∗ *pph*, ppl_const_ConSys_t *cs*)

Builds a new NNC polyhedron from the system of constraints `cs` and writes an handle for the newly created polyhedron at address `pph`.

The new polyhedron will inherit the space dimension of `cs`.

7.3.5.8 int ppl_new_NNC_Polyhedron_recycle_ConSys (ppl_Polyhedron_t ∗ *pph*, ppl_ConSys_t *cs*)

Builds a new NNC polyhedron recycling the system of constraints `cs` and writes an handle for the newly created polyhedron at address `pph`.

Since `cs` will be *the* system of constraints of the new polyhedron, the space dimension is also inherited.

Warning:

This function modifies the constraint system referenced by `cs:` upon return, no assumption can be made on its value.

7.3.5.9 int ppl_new_C_Polyhedron_from_GenSys (ppl_Polyhedron_t ∗ *pph*, ppl_const_GenSys_t *gs*)

Builds a new closed polyhedron from the system of generators `gs` and writes an handle for the newly created polyhedron at address `pph`.

The new polyhedron will inherit the space dimension of `gs`.

7.3.5.10 int ppl_new_C_Polyhedron_recycle_GenSys (ppl_Polyhedron_t ∗ *pph*, ppl_GenSys_t *gs*)

Builds a new closed polyhedron recycling the system of generators `gs` and writes an handle for the newly created polyhedron at address `pph`.

Since `gs` will be *the* system of generators of the new polyhedron, the space dimension is also inherited.

Warning:
This function modifies the generator system referenced by `gs`: upon return, no assumption can be made on its value.

7.3.5.11 int ppl_new_NNC_Polyhedron_from_GenSys (ppl_Polyhedron_t ∗ *pph*, ppl_const_GenSys_t *gs*)

Builds a new NNC polyhedron from the system of generators `gs` and writes an handle for the newly created polyhedron at address `pph`.

The new polyhedron will inherit the space dimension of `gs`.

7.3.5.12 int ppl_new_NNC_Polyhedron_recycle_GenSys (ppl_Polyhedron_t ∗ *pph*, ppl_GenSys_t *gs*)

Builds a new NNC polyhedron recycling the system of generators `gs` and writes an handle for the newly created polyhedron at address `pph`.

Since `gs` will be *the* system of generators of the new polyhedron, the space dimension is also inherited.

Warning:
This function modifies the generator system referenced by `gs`: upon return, no assumption can be made on its value.

7.3.5.13 int ppl_new_C_Polyhedron_from_bounding_box (ppl_Polyhedron_t ∗ *pph*, ppl_dimension_type(∗ *space_dimension*)(void), int(∗ *is_empty*)(void), int(∗ *get_lower_bound*)(ppl_dimension_type k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d), int(∗ *get_upper_bound*)(ppl_dimension_type k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d))

Builds a new C polyhedron corresponding to an interval-based bounding box, writing a handle for the newly created polyhedron at address `pph`.

If an interval of the bounding box is provided with any finite but open bound, then the polyhedron is not built and the value `PPL_ERROR_INVALID_ARGUMENT` is returned. The bounding box is accessed by using the following functions, passed as arguments:

```
ppl_dimension_type space_dimension()
```

returns the dimension of the vector space enclosing the polyhedron represented by the bounding box.

```
int is_empty()
```

returns 0 if and only if the bounding box describes a non-empty set. The function `is_empty()` will always be called before the other functions. However, if `is_empty()` does not return 0, none of the functions below will be called.

```
int get_lower_bound(ppl_dimension_type k, int closed,
                    ppl_Coefficient_t n, ppl_Coefficient_t d)
```

Let I the interval corresponding to the `k`-th dimension. If I is not bounded from below, simply return 0. Otherwise, set `closed`, `n` and `d` as follows: `closed` is set to 0 if the lower boundary of I is open and is set to a value different from zero otherwise; `n` and `d` are assigned the integers n and d such that the canonical fraction n/d corresponds to the greatest lower bound of I. The fraction n/d is in canonical form if and only if n and d have no common factors and d is positive, $0/1$ being the unique representation for zero.

```
int get_upper_bound(ppl_dimension_type k, int closed,
                    ppl_Coefficient_t n, ppl_Coefficient_t d)
```

Let I the interval corresponding to the `k`-th dimension. If I is not bounded from above, simply return 0. Otherwise, set `closed`, `n` and `d` as follows: `closed` is set to 0 if the upper boundary of I is open and is set to a value different from 0 otherwise; `n` and `d` are assigned the integers n and d such that the canonical fraction n/d corresponds to the least upper bound of I.

7.3.5.14 int ppl_new_NNC_Polyhedron_from_bounding_box (ppl_Polyhedron_t $*$ *pph*, ppl_dimension_type($*$ *space_dimension*)(void), int($*$ *is_empty*)(void), int($*$ *get_lower_bound*)(ppl_dimension_type k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d), int($*$ *get_upper_bound*)(ppl_dimension_type k, int closed, ppl_Coefficient_t n, ppl_Coefficient_t d))

Builds a new C polyhedron corresponding to an interval-based bounding box, writing a handle for the newly created polyhedron at address `pph`.

The bounding box is accessed by using the following functions, passed as arguments:

```
ppl_dimension_type space_dimension()
```

returns the dimension of the vector space enclosing the polyhedron represented by the bounding box.

```
int is_empty()
```

returns 0 if and only if the bounding box describes a non-empty set. The function `is_empty()` will always be called before the other functions. However, if `is_empty()` does not return 0, none of the functions below will be called.

```
int get_lower_bound(ppl_dimension_type k, int closed,
                    ppl_Coefficient_t n, ppl_Coefficient_t d)
```

Let I the interval corresponding to the `k`-th dimension. If I is not bounded from below, simply return 0. Otherwise, set `closed`, `n` and `d` as follows: `closed` is set to 0 if the lower boundary of I is open and is set to a value different from zero otherwise; `n` and `d` are assigned the integers n and d such that the canonical fraction n/d corresponds to the greatest lower bound of I. The fraction n/d is in canonical form if and only if n and d have no common factors and d is positive, $0/1$ being the unique representation for zero.

```
int get_upper_bound(ppl_dimension_type k, int closed,
                    ppl_Coefficient_t n, ppl_Coefficient_t d)
```

Let I the interval corresponding to the `k`-th dimension. If I is not bounded from above, simply return 0. Otherwise, set `closed`, `n` and `d` as follows: `closed` is set to 0 if the upper boundary of I is open and is set to a value different from 0 otherwise; `n` and `d` are assigned the integers n and d such that the canonical fraction n/d corresponds to the least upper bound of I.

7.3.5.15 int ppl_Polyhedron_relation_with_Constraint (ppl_const_Polyhedron_t *ph*, ppl_const_-Constraint_t *c*)

Checks the relation between the polyhedron `ph` with the constraint `c`.

If successful, returns a non-negative integer that is obtained as the bitwise or of the bits (chosen among PPL_POLY_CON_RELATION_IS_DISJOINT PPL_POLY_CON_RELATION_STRICTLY_-INTERSECTS, PPL_POLY_CON_RELATION_IS_INCLUDED, and PPL_POLY_CON_RELATION_-SATURATES) that describe the relation between `ph` and `c`.

7.3.5.16 int ppl_Polyhedron_relation_with_Generator (ppl_const_Polyhedron_t *ph*, ppl_const_-Generator_t *g*)

Checks the relation between the polyhedron `ph` with the generator `g`.

If successful, returns a non-negative integer that is obtained as the bitwise or of the bits (only PPL_POLY_-GEN_RELATION_SUBSUMES, at present) that describe the relation between `ph` and `g`.

7.3.5.17 int ppl_Polyhedron_shrink_bounding_box (ppl_const_Polyhedron_t *ph*, unsigned int *complexity*, void($*$ *set_empty*)(void), void($*$ *raise_lower_bound*)(ppl_dimension_type k, int closed, ppl_-const_Coefficient_t n, ppl_const_Coefficient_t d), void($*$ *lower_upper_bound*)(ppl_dimension_type k, int closed, ppl_const_Coefficient_t n, ppl_const_Coefficient_t d))

Use `ph` to shrink a generic, interval-based bounding box. The bounding box is abstractly provided by means of the parameters.

Parameters:

ph The polyhedron that is used to shrink the bounding box;

complexity The code of the complexity class of the algorithm to be used. Must be one of PPL_COMPLEXITY_CLASS_POLYNOMIAL, PPL_COMPLEXITY_CLASS_SIMPLEX, or PPL_COMPLEXITY_CLASS_ANY;

set_empty A pointer to a void function with no arguments that causes the bounding box to become empty, i.e., to represent the empty set;

raise_lower_bound A pointer to a void function with arguments `(ppl_dimension_type k, int closed, ppl_const_Coefficient_t n, ppl_const_Coefficient_t d)` that intersects the interval corresponding to the `k`-th dimension with $[n/d, +\infty)$ if `closed` is non-zero, with $(n/d, +\infty)$ if `closed` is zero. The fraction n/d is in canonical form, that is, n and d have no common factors and d is positive, $0/1$ being the unique representation for zero;

lower_upper_bound a pointer to a void function with argument `(ppl_dimension_type k, int closed, ppl_const_Coefficient_t n, ppl_const_Coefficient_t d)` that intersects the interval corresponding to the `k`-th dimension with $(-\infty, n/d]$ if `closed` is non-zero, with $(-\infty, n/d)$ if `closed` is zero. The fraction n/d is in canonical form.

7.3.5.18 int ppl_Polyhedron_maximize (ppl_const_Polyhedron_t *ph*, ppl_const_LinExpression_t *le*, ppl_Coefficient_t *sup_n*, ppl_Coefficient_t *sup_d*, int ∗ *pmaximum*, ppl_const_Generator_t ∗ *ppoint*)

Returns a positive integer if `ph` is not empty and `le` is bounded from above in `ph`, in which case the supremum value and a point where `le` reaches it are computed.

Parameters:

ph The polyhedron constraining `le`;

le The linear expression to be maximized subject to `ph`;

sup_n Will be assigned the numerator of the supremum value;

sup_d Will be assigned the denominator of the supremum value;

pmaximum Will store 1 in this location if the supremum is also the maximum, will store 0 otherwise;

ppoint When nonzero, a point or closure point where `le` reaches the extremum value will be stored here. If `ph` is empty or `le` is not bounded from above, 0 is returned and `sup_n`, `sup_d`, `*pmaximum` and `*ppoint` are left untouched.

7.3.5.19 int ppl_Polyhedron_minimize (ppl_const_Polyhedron_t *ph*, ppl_const_LinExpression_t *le*, ppl_Coefficient_t *inf_n*, ppl_Coefficient_t *inf_d*, int ∗ *pminimum*, ppl_const_Generator_t ∗ *ppoint*)

Returns a positive integer if `ph` is not empty and `le` is bounded from above in `ph`, in which case the infimum value and a point where `le` reaches it are computed.

Parameters:

ph The polyhedron constraining `le`;

le The linear expression to be minimized subject to `ph`;

inf_n Will be assigned the numerator of the infimum value;

inf_d Will be assigned the denominator of the infimum value;

pminimum Will store 1 in this location if the infimum is also the minimum, will store 0 otherwise;

ppoint When nonzero, a point or closure point where `le` reaches the extremum value will be stored here. If `ph` is empty or `le` is not bounded from below, 0 is returned and `inf_n`, `inf_d`, `*pminimum` and `*ppoint` are left untouched.

7.3.5.20 int ppl_Polyhedron_equals_Polyhedron (ppl_const_Polyhedron_t *x*, ppl_const_Polyhedron_t *y*)

Returns a positive integer if `x` and `y` are the same polyhedron; return 0 if they are different.

Note that `x` and `y` may be topology- and/or dimension-incompatible polyhedra: in those cases, the value 0 is returned.

7.3.5.21 int ppl_Polyhedron_add_recycled_constraints (ppl_Polyhedron_t *ph*, ppl_ConSys_t *cs*)

Adds the system of constraints `cs` to the system of constraints of `ph`.

Warning:

This function modifies the constraint system referenced by `cs:` upon return, no assumption can be made on its value.

7.3.5.22 int ppl_Polyhedron_add_recycled_constraints_and_minimize (ppl_Polyhedron_t *ph*, ppl_-ConSys_t *cs*)

Adds the system of constraints `cs` to the system of constraints of `ph`. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return, `ph` is guaranteed to be minimized.

Warning:
This function modifies the constraint system referenced by `cs`: upon return, no assumption can be made on its value.

7.3.5.23 int ppl_Polyhedron_add_recycled_generators (ppl_Polyhedron_t *ph*, ppl_GenSys_t *gs*)

Adds the system of generators `gs` to the system of generators of `ph`.

Warning:
This function modifies the generator system referenced by `gs`: upon return, no assumption can be made on its value.

7.3.5.24 int ppl_Polyhedron_add_recycled_generators_and_minimize (ppl_Polyhedron_t *ph*, ppl_-GenSys_t *gs*)

Adds the system of generators `gs` to the system of generators of `ph`. Returns a positive integer if the resulting polyhedron is non-empty; returns 0 if it is empty. Upon successful return, `ph` is guaranteed to be minimized.

Warning:
This function modifies the generator system referenced by `gs`: upon return, no assumption can be made on its value.

7.3.5.25 int ppl_Polyhedron_affine_image (ppl_Polyhedron_t *ph*, ppl_dimension_type *var*, ppl_-const_LinExpression_t *le*, ppl_const_Coefficient_t *d*)

Transforms the polyhedron `ph`, assigning an affine expression to the specified variable.

Parameters:
- ***ph*** The polyhedron that is transformed;
- ***var*** The variable to which the affine expression is assigned;
- ***le*** The numerator of the affine expression;
- ***d*** The denominator of the affine expression.

7.3.5.26 int ppl_Polyhedron_affine_preimage (ppl_Polyhedron_t *ph*, ppl_dimension_type *var*, ppl_-const_LinExpression_t *le*, ppl_const_Coefficient_t *d*)

Transforms the polyhedron `ph`, substituting an affine expression to the specified variable.

Parameters:
- ***ph*** The polyhedron that is transformed;
- ***var*** The variable to which the affine expression is substituted;
- ***le*** The numerator of the affine expression;
- ***d*** The denominator of the affine expression.

7.3.5.27 int ppl_Polyhedron_generalized_affine_image (ppl_Polyhedron_t *ph*, ppl_dimension_type *var*, enum ppl_enum_Constraint_Type *relsym*, ppl_const_LinExpression_t *le*, ppl_const_Coefficient_t *d*)

Assigns to `ph` the image of `ph` with respect to the **generalized affine transfer function**(p. 9) $\mathrm{var}' \bowtie \frac{\mathrm{expr}}{\mathrm{denominator}}$, where $\bowtie$ is the relation symbol encoded by `relsym`.

Parameters:

ph The polyhedron that is transformed;

var The left hand side variable of the generalized affine transfer function;

relsym The relation symbol;

le The numerator of the right hand side affine expression;

d The denominator of the right hand side affine expression.

7.3.5.28 int ppl_Polyhedron_generalized_affine_image_lhs_rhs (ppl_Polyhedron_t *ph*, ppl_const_LinExpression_t *lhs*, enum ppl_enum_Constraint_Type *relsym*, ppl_const_LinExpression_t *rhs*)

Assigns to `ph` the image of `ph` with respect to the **generalized affine transfer function**(p. 9) $\mathrm{lhs}' \bowtie \mathrm{rhs}$, where $\bowtie$ is the relation symbol encoded by `relsym`.

Parameters:

ph The polyhedron that is transformed;

lhs The left hand side affine expression;

relsym The relation symbol;

rhs The right hand side affine expression.

7.3.5.29 int ppl_Polyhedron_map_dimensions (ppl_Polyhedron_t *ph*, ppl_dimension_type *maps*[ ], size_t *n*)

Remaps the dimensions of the vector space according to a **partial function**(p. 7). This function is specified by means of the `maps` array, which has `n` entries.

The partial function is defined on dimension `i` if `i < n` and `maps[i] != ppl_not_a_dimension`; otherwise it is undefined on dimension `i`. If the function is defined on dimension `i`, then dimension `i` is mapped onto dimension `maps[i]`.

The result is undefined if `maps` does not encode a partial function with the properties described in the **specification of the mapping operator**(p. 7).

7.4 Prolog Language Interface

The Parma Polyhedra Library comes equipped with a Prolog interface. Despite the lack of standardization of Prolog's foreign language interfaces, the PPL Prolog interface supports several Prolog systems and, to the extent this is possible, provides a uniform view of the library from each such systems.

The system-independent features of the library are described in Section **System-Independent Features**(p. 49). Section **Compilation and Installation**(p. 65) explains how the various incarnations of the Prolog interface are compiled and installed. Section **System-Dependent Features**(p. 65) illustrates the system-dependent features of the interface for all the supported systems.

System-Independent Features

The Prolog interface provides access to the PPL polyhedra. A general introduction to convex polyhedra, their representation in the PPL and the operations provided by the PPL is given in Sections **The Main Features**(p. 1), **Convex Polyhedra**(p. 2), **Representations of Convex Polyhedra**(p. 3) and **Operations on Convex Polyhedra**(p. 6) of this manual. Here we just describe those aspects that are specific to the Prolog interface.

Overview First, here is a list of notes with general information and advice on the use of the interface.

- The Prolog interface to the PPL is initialized and finalized by the predicates `ppl_initialize/0` and `ppl_finalize/0`. Thus the only interface predicates callable after `ppl_finalize/0` are `ppl_finalize/0` itself (this further call has no effect) and `ppl_initialize/0`, after which the interface's services are usable again. Some Prolog systems allow the specification of initialization and deinitialization functions in their foreign language interfaces. The corresponding incarnations of the PPL-Prolog interface have been written so that `ppl_initialize/0` and/or `ppl_finalize/0` are called automatically. Section **System-Dependent Features**(p. 65) will detail in which cases initialization and finalization is automatically performed or is left to the Prolog programmer's responsibility. However, for portable applications, it is best to invoke `ppl_initialize/0` and `ppl_finalize/0` explicitly: since they can be called multiple times without problems, this will result in enhanced portability at a cost that is, by all means, negligible.

- A PPL polyhedron can only be accessed by means of a Prolog term called a *handle*. Note, however, that the data structure of a handle, is implementation-dependent, system-dependent and version-dependent, and, for this reason, deliberately left unspecified. What we do guarantee is that the handle requires very little memory.

- A Prolog term can be bound to a valid handle by using:

```
ppl_new_Polyhedron_from_dimension/3,
ppl_new_Polyhedron_empty_from_dimension/3,
ppl_new_Polyhedron_from_Polyhedron/4,
ppl_new_Polyhedron_from_constraints/3,
ppl_new_Polyhedron_from_generators/3.
ppl_new_Polyhedron_from_bounding_box/3.
```

  These predicates will create or copy a PPL polyhedron and construct a valid handle for referencing it. The first argument (in the case of `ppl_new_Polyhedron_from_Polyhedron/4`, the first and third arguments) denotes the topology and can be either `c` or `nnc` indicating a C or NNC polyhedron, respectively. The third argument (in the case of `ppl_new_Polyhedron_from_Polyhedron/4`, the fourth argument) is a Prolog term that is unified with a new valid handle for accessing this polyhedron.

- As soon as a PPL polyhedron is no longer required, the memory occupied by it should be released using the PPL predicate `ppl_delete_Polyhedron/1`. To understand why this is important, consider a Prolog program and a variable that is bound to a Herbrand term. When the variable dies (goes out of scope) or is uninstantiated (on backtracking) the term it is bound to is amenable to garbage collection. But this only applies for the standard domain of the language: Herbrand terms. In Prolog+PPL, when a variable bound to a handle for a PPL Polyhedron dies or is uninstantiated, the handle can be garbage-collected, but the polyhedra to which the handle refers will not be released. Once a handle has been used as an argument in `ppl_delete_Polyhedron/1`, it becomes invalid.

- For a PPL polyhedron with space dimension k, the identifiers used for the PPL variables must lie between 0 and $k-1$ and correspond to the indices of the associated Cartesian axes. When using the predicates that combine PPL polyhedra or add constraints or generators to a representation of a PPL polyhedron, the polyhedra referenced and any constraints or generators in the call should follow all the space dimension-compatibility rules stated in Section **Representations of Convex Polyhedra**(p. 3).

- As explained above, a polyhedron has a fixed topology C or NNC, that is determined at the time of its initialization. All subsequent operations on the polyhedron must respect all the topological compatibility rules stated in Section **Representations of Convex Polyhedra**(p. 3).

- Any application using the PPL should make sure that only the intended version(s) of the library are ever used. Predicates

```
ppl_version_major/1,
ppl_version_minor/1,
ppl_version_revision/1,
ppl_version_beta/1,
ppl_version/1,
ppl_banner.
```

  allow run-time checking of information about the version being used.

PPL Predicate List Here is a list of all the PPL predicates provided by the Prolog interface.

`ppl_version_major(?Integer)`

`ppl_version_minor(?Integer)`

`ppl_version_revision(?Integer)`

`ppl_version_beta(?Integer)`

`ppl_version(?Atom)`

`ppl_banner(?Atom)`

`ppl_max_space_dimension(?Integer)`

`ppl_initialize`

`ppl_finalize`

`ppl_set_timeout_exception_atom(+Atom)`

`ppl_set_timeout(+Integer)`

`ppl_reset_timeout`

`ppl_new_Polyhedron_from_dimension(+Topology, +Integer, -Handle)`

`ppl_new_Polyhedron_empty_from_dimension(+Topology, +Integer, -Handle)`

`ppl_new_Polyhedron_from_Polyhedron(+Topology_1, +Handle_1, +Topology_2, -Handle_2)`

`ppl_new_Polyhedron_from_constraints(+Topology, +Constraint_System, -Handle)`

`ppl_new_Polyhedron_from_generators(+Topology, +Generator_System, -Handle)`

`ppl_new_Polyhedron_from_bounding_box(+Topology, +Box, -Handle)` ppl_Polyhedron_swap(+Handle1, +Handle2)

ppl_delete_Polyhedron(+Handle)

ppl_Polyhedron_space_dimension(+Handle, -Integer)

ppl_Polyhedron_get_constraints(+Handle, -Constraint_System)

ppl_Polyhedron_get_minimized_constraints(+Handle, -Constraint_System)

ppl_Polyhedron_get_generators(+Handle, -Generator_System)

ppl_Polyhedron_get_minimized_generators(+Handle, -Generator_System)

ppl_Polyhedron_relation_with_constraint(+Handle, +Constraint, -Relation)

ppl_Polyhedron_relation_with_generator(+Handle, +Generator, -Relation)

ppl_Polyhedron_get_bounding_box(+Handle, +Complexity, -Box)

ppl_Polyhedron_is_empty(+Handle)

ppl_Polyhedron_is_universe(+Handle)

ppl_Polyhedron_is_bounded(+Handle)

ppl_Polyhedron_bounds_from_above(+Handle, +LinExpr)

ppl_Polyhedron_bounds_from_below(+Handle, +LinExpr)

ppl_Polyhedron_maximize(+Handle, +LinExpr, ?Integer, ?Integer, ?Bool)

ppl_Polyhedron_maximize_with_point(+Handle, +LinExpr, ?Integer, ?Integer, ?Bool, ?Point)

ppl_Polyhedron_minimize(+Handle, +LinExpr, ?Integer, ?Integer, ?Bool)

ppl_Polyhedron_minimize_with_point(+Handle, +LinExpr, ?Integer, ?Integer, ?Bool, ?Point)

ppl_Polyhedron_is_topologically_closed(+Handle)

ppl_Polyhedron_contains_Polyhedron(+Handle_1, +Handle_2)

ppl_Polyhedron_strictly_contains_Polyhedron(+Handle_1, +Handle_2)

ppl_Polyhedron_is_disjoint_from_Polyhedron(+Handle_1, +Handle_2)

ppl_Polyhedron_equals_Polyhedron(+Handle_1, +Handle_2)

ppl_Polyhedron_OK(+Handle)

ppl_Polyhedron_add_constraint(+Handle, +Constraint)

ppl_Polyhedron_add_constraint_and_minimize(+Handle, +Constraint)

ppl_Polyhedron_add_generator(+Handle, +Generator)

ppl_Polyhedron_add_generator_and_minimize(+Handle, +Generator)

ppl_Polyhedron_add_constraints(+Handle, +Constraint_System)

ppl_Polyhedron_add_constraints_and_minimize(+Handle, +Constraint_System)

ppl_Polyhedron_add_generators(+Handle, +Generator_System) ppl_Polyhedron_add_generators_and_minimize(+Handle, +Generator_-System)

ppl_Polyhedron_intersection_assign(+Handle_1, +Handle_2)

ppl_Polyhedron_intersection_assign_and_minimize(+Handle_1, +Handle_2)

ppl_Polyhedron_poly_hull_assign(+Handle_1, +Handle_2)

ppl_Polyhedron_poly_hull_assign_and_minimize(+Handle_1, +Handle_2)

ppl_Polyhedron_poly_difference_assign(+Handle_1, +Handle_2)

ppl_Polyhedron_affine_image(+Handle, +PPL_Var, +LinExpr, +Integer)

ppl_Polyhedron_affine_preimage(+Handle, +PPL_Var, +LinExpr, +Integer)

ppl_Polyhedron_generalized_affine_image(+Handle, +PPL_Var, +Relation_-Symbol, +LinExpr, +Integer)

ppl_Polyhedron_generalized_affine_image_lhs_rhs(+Handle, +LinExpr1, +Relation_Symbol, +LinExpr2)

ppl_Polyhedron_time_elapse_assign(+Handle_1, +Handle_2)

ppl_Polyhedron_BHRZ03_widening_assign_with_token(+Handle_1, +Handle_2, ?Integer)

ppl_Polyhedron_BHRZ03_widening_assign(+Handle_1, +Handle_2)

ppl_Polyhedron_limited_BHRZ03_extrapolation_assign_with_-token(+Handle_1, +Handle_2, +Constraint_System, ?Integer)

ppl_Polyhedron_limited_BHRZ03_extrapolation_assign(+Handle_1, +Handle_2, +Constraint_System)

ppl_Polyhedron_bounded_BHRZ03_extrapolation_assign_with_-token(+Handle_1, +Handle_2, +Constraint_System, ?Integer)

ppl_Polyhedron_bounded_BHRZ03_extrapolation_assign(+Handle_1, +Handle_2, +Constraint_System)

ppl_Polyhedron_H79_widening_assign_with_token(+Handle_1, +Handle_2, ?Integer)

ppl_Polyhedron_H79_widening_assign(+Handle_1, +Handle_2)

ppl_Polyhedron_limited_H79_extrapolation_assign_with_token(+Handle_1, +Handle_2, +Constraint_System, ?Integer)

ppl_Polyhedron_limited_H79_extrapolation_assign(+Handle_1, +Handle_2, +Constraint_System)

ppl_Polyhedron_bounded_H79_extrapolation_assign_with_token(+Handle_1, +Handle_2, +Constraint_System)

ppl_Polyhedron_bounded_H79_extrapolation_assign(+Handle_1, +Handle_2, +Constraint_System, ?Integer)

ppl_Polyhedron_topological_closure_assign(+Handle)

ppl_Polyhedron_add_dimensions_and_embed(+Handle, +Integer)

ppl_Polyhedron_add_dimensions_and_project(+Handle, +Integer)

ppl_Polyhedron_concatenate_assign(+Handle1, +Handle2)

ppl_Polyhedron_remove_dimensions(+Handle, +List_of_PPL_Vars)

```
ppl_Polyhedron_remove_higher_dimensions(+Handle, +Integer))
ppl_Polyhedron_expand_dimension(+Handle, +PPL_Var, +Integer))
ppl_Polyhedron_fold_dimensions(+Handle, +List_of_PPL_Vars, +PPL_Var))
ppl_Polyhedron_map_dimensions(+Handle, +P_Func))
```

PPL Predicate Specifications The PPL predicates provided by the Prolog interface are specified below. The specification uses the following grammar rules:

```
Handle      --> Prolog term

Topology    --> c | nnc

VarId       --> number | + number      variable identifier

PPL_Var     --> '$VAR'(VarId)          PPL variable

LinExpr     --> PPL_Var                PPL variable
             | number
             | + LinExpr               unary plus
             | - LinExpr               unary minus
             | LinExpr + LinExpr       addition
             | LinExpr - LinExpr       subtraction
             | number * LinExpr        multiplication
             | LinExpr * number        multiplication

Relation_Symbol
            --> =                      equals
             | =<                      less than or equal
             | >=                      greater than or equal
             | <                       strictly less than
             | >                       strictly greater than

Denominator --> number
             | + number | - number     number must be non-zero

Constraint  --> LinExpr Relation_Symbol LinExpr
                                       constraint

Constraint_System                      list of constraints
            --> []
             | [Constraint | Constraint_System]

Generator   --> point(LinExpr)         point
             | point(LinExpr, Denominator)
                                       point
             | closure_point(LinExpr)  closure point
             | closure_point(LinExpr, Denominator)
                                       closure point
              (the point or closure point is defined by LinExpr/Denominator.)
             | ray(LinExpr)            ray
             | line(LinExpr)           line

Generator_System                       list of generators
            --> []
             | [Generator | Generator_System]

Atom        --> Prolog atom

Relation    --> is_disjoint            between a constraint and a polyhedron
             | strictly_intersects     between a constraint and a polyhedron
             | is_included             between a constraint and a polyhedron
```

```
            | saturates                 between a constraint and a polyhedron
            | subsumes                  between a generator and a polyhedron

Relation_List                           list of relations
             --> []
            | [Relation | Relation_List]


Complexity  --> polynomial | simplex | any

Rational_Numerator
             --> number | + number | - number

Rational_Denominator
             --> number                 number must be non-zero

Rational     --> Rational_Numerator     rational number
            | Rational_Numerator/Rational_Denominator

Bound        --> c(Rational)            closed rational limit
            | o(Rational)               open rational limit
            | o(pinf)                   unbounded in the positive direction
            | o(minf)                   unbounded in the negative direction

Interval     --> i(Bound, Bound)        rational interval

Box          --> []                     list of intervals
            | [Interval | Box]

Vars_Pair    --> PPLVar - PPLVar        map relation

P_Func       --> []                     list of map relations
            | [Vars_Pair | P_Func].
```

Below is a short description of each of the interface predicates. For full definitions of terminology used here, see Sections **The Main Features**(p. 1), **Convex Polyhedra**(p. 2), **Representations of Convex Polyhedra**(p. 3) and **Operations on Convex Polyhedra**(p. 6) of this manual.

`ppl_version_major(?Integer)` Unifies `Integer` with the major number of the PPL version.

`ppl_version_minor(?Integer)` Unifies `Integer` with the minor number of the PPL version.

`ppl_version_revision(?Integer)` Unifies `Integer` with the revision number of the PPL version.

`ppl_version_beta(?Integer)` Unifies `Integer` with the beta number of the PPL version.

`ppl_version(?Atom)` Unifies `Atom` with the PPL version.

`ppl_banner(?Atom)` Unifies `Atom` with information about the PPL version, the licensing, the lack of any warranty whatsoever, the C++ compiler used to build the library, where to report bugs and where to look for further information.

`ppl_max_space_dimension(?Integer)` Unifies `Integer` with the maximum space dimension this library can handle.

`ppl_initialize` Initializes the PPL interface. Multiple calls to `ppl_initialize` does no harm.

`ppl_finalize` Finalizes the PPL interface. Once this is executed, the next call to an interface predicate must either be to `ppl_initialize` or to `ppl_finalize`. Multiple calls to `ppl_finalize` does no harm.

`ppl_set_timeout_exception_atom(+Atom)` Sets the atom to be thrown by timeout exceptions to `Atom`. The default value is `time_out`.

`ppl_timeout_exception_atom(?Atom)` The atom to be thrown by timeout exceptions is unified with `Atom`.

`ppl_set_timeout(+Integer)` Computations taking exponential time will be interrupted some time after `Integer` ms after that call. If the computation is interrupted that way, the current timeout exception atom will be thrown. `Integer` must be strictly greater than zero.

`ppl_reset_timeout` Resets the timeout time so that the computation is not interrupted.

`ppl_new_Polyhedron_from_dimension(+Topology, +Integer, -Handle)` Creates a new universe C or NNC polyhedron $\mathcal{P}$, depending on the value of `Topology`, with `Integer` dimensions. `Handle` is unified with the handle for $\mathcal{P}$. Thus the query

```
?- ppl_new_Polyhedron_from_dimension(c, 3, X).
```

creates the C polyhedron defining the 3-dimensional vector space $\mathbb{R}^3$ with `X` bound to a valid handle for accessing it.

`ppl_new_Polyhedron_empty_from_dimension(+Topology, +Integer, -Handle)` Creates a new empty C or NNC polyhedron $\mathcal{P}$, depending on the value of `Topology`, with `Integer` dimensions. `Handle` is unified with the handle for $\mathcal{P}$. Thus the query

```
?- ppl_new_Polyhedron_empty_from_dimension(nnc, 3, X).
```

creates an empty NNC polyhedron embedded in $\mathbb{R}^3$ with `X` bound to a valid handle for accessing it.

`ppl_new_Polyhedron_from_Polyhedron(+Topology_1, +Handle_1, +Topology_2, -Handle_2)` If `Handle_1` refers to a C or NNC polyhedron $\mathcal{P}_1$ (depending on the value of `Topology_1`), then this creates a copy $\mathcal{P}_2$ of $\mathcal{P}_1$ with topology C or NNC, depending on the value of `Topology_2`. `Handle_2` is unified with the handle for $\mathcal{P}_2$. Thus the query

```
?- ppl_new_Polyhedron_empty_from_dimension(nnc, 3, X),
   ppl_new_Polyhedron_from_Polyhedron(c, X, nnc, Y).
```

creates an empty C polyhedron embedded in $\mathbb{R}^3$ referenced by `X` and then makes a copy, converting the topology to an NNC polyhedron. with `Y` bound to a valid handle for accessing it.

When using `ppl_new_Polyhedron_from_Polyhedron/2`, when the source polyhedron is NNC and the copy is C, care must be taken that the source polyhedron referenced by `Handle1` is topologically closed.

`ppl_new_Polyhedron_from_constraints(+Topology, +Constraint_System, -Handle)` Creates a polyhedron $\mathcal{P}$ represented by `Constraint_System` with topology C or NNC, depending on the value of `Topology`. `Handle` is unified with the handle for $\mathcal{P}$.

`ppl_new_Polyhedron_from_generators(+Topology, +Generator_System, -Handle)` Creates a polyhedron $\mathcal{P}$ represented by `Generator_System` with topology C or NNC, depending on the value of `Topology`. `Handle` is unified with the handle for $\mathcal{P}$.

`ppl_new_Polyhedron_from_bounding_box(+Topology, +Box, -Handle)` Creates a polyhedron $\mathcal{P}$ represented by `Box` with topology C or NNC, depending on the value of `Topology`, and `Handle` is unified with the handle for $\mathcal{P}$. A bound of the form `o(Rational)` can be included in an interval in `Box` only if `Topology` is `nnc`.

`ppl_Polyhedron_swap(+Handle1, +Handle2)` Swaps the polyhedron referenced by `Handle1` with the one referenced by `Handle2`. The polyhedra $\mathcal{P}$ and $\mathcal{Q}$ must have the same topology.

`ppl_delete_Polyhedron(+Handle)` Deletes the polyhedron referenced by `Handle`. After execution, `Handle` is no longer a valid handle for a PPL polyhedron.

`ppl_Polyhedron_space_dimension(+Handle, ?Integer)` Unifies the space dimension of the polyhedron referenced by `Handle` with `Integer`.

`ppl_Polyhedron_get_constraints(+Handle, ?Constraint_System)` Unifies `Constraint_System` with a list of the constraints in the constraints system representing the polyhedron referenced by `Handle`.

`ppl_Polyhedron_get_minimized_constraints(+Handle, ?Constraint_System)` Unifies `Constraint_System` with a minimized list of the constraints in the constraints system representing the polyhedron referenced by `Handle`.

`ppl_Polyhedron_get_generators(+Handle, ?Generator_System)` Unifies `Generator_System` with a list of the generators in the generators system representing the polyhedron referenced by `Handle`.

`ppl_Polyhedron_get_minimized_generators(+Handle, ?Generator_System)` Unifies `Generator_System` with a minimized list of the generators in the generators system representing the polyhedron referenced by `Handle`.

`ppl_Polyhedron_relation_with_constraint(+Handle, +Constraint, ?Relation_List)` Unifies `Relation_List` with the list of relations the polyhedron referenced by `Handle` has with `Constraint`. The possible relations are listed in the grammar rules above; their meaning is given in the paragraph **specifying the relation_with operations**(p. 10) in Section **Operations on Convex Polyhedra**(p. 6).

ppl_Polyhedron_relation_with_generator(+Handle, +Generator, ?Relation_List) Unifies Relation_List with the list of relations the polyhedron referenced by Handle has with Generator. The possible relations are listed in the grammar rules above; their meaning is given in the paragraph **specifying the relation_with operations**(p. 10) in Section **Operations on Convex Polyhedra**(p. 6).

ppl_Polyhedron_get_bounding_box(+Handle, +Complexity, ?Box) Succeeds if and only if the bounding box of the polyhedron referenced by Handle unifies with the box defined by Box. E.g.,

```
?- A = '$VAR'(0), B = '$VAR'(1),
   ppl_new_Polyhedron_from_constraints(nnc, [B > 0, 4*A =< 2], X),
   ppl_Polyhedron_get_bounding_box(X, any, Box).

Box = [i(o(minf), c(1/2)), i(o(0), o(pinf))].
```

Note that the rational numbers in Box are in canonical form. E.g., the following will fail:

```
?- A = '$VAR'(0), B = '$VAR'(1),
   ppl_new_Polyhedron_from_constraints(nnc, [B > 0, 4*A =< 2], X),
   ppl_Polyhedron_get_bounding_box(X, any, Box),
   Box = [i(o(minf), c(2/4)), i(o(0), o(pinf))].
```

The complexity class Complexity determining the algorithm to be used has the following meaning:

- polynomial allows code of the worst-case polynomial complexity class;
- simplex allows code of the worst-case exponential but typically polynomial complexity class;
- any allows code of the universal complexity class.

ppl_Polyhedron_is_empty(+Handle) Succeeds if and only if the polyhedron referenced by Handle is empty.

ppl_Polyhedron_is_universe(+Handle) Succeeds if and only if the polyhedron referenced by Handle is the universe.

ppl_Polyhedron_is_bounded(+Handle) Succeeds if and only if the polyhedron referenced by Handle is bounded.

ppl_Polyhedron_bounds_from_above(+Handle, +LinExpr) Succeeds if and only if LinExpr is bounded from above in the polyhedron referenced by Handle.

ppl_Polyhedron_bounds_from_below(+Handle, +LinExpr) Succeeds if and only if LinExpr is bounded from below in the polyhedron referenced by Handle.

ppl_Polyhedron_maximize(+Handle, +LinExpr, ?Integer, ?Integer, ?Bool) Succeeds if and only if the polyhedron P referenced by Handle is not empty and LinExpr is bounded from above in P.

Integer1 is unified with the numerator of the supremum value and Integer2 with the denominator of the supremum value. If the supremum is also the maximum, Bool is unified with the atom true and, otherwise, unified with the atom false.

`ppl_Polyhedron_maximize_with_point(+Handle, +LinExpr, ?Integer1, ?Integer2, ?Bool, ?Point)` Succeeds if and only if the polyhedron P referenced by `Handle` is not empty and `LinExpr` is bounded from above in P.

`Integer1` is unified with the numerator of the supremum value, `Integer2` with the denominator of the supremum value, and `Point` with a point or closure point where `LinExpr` reaches this value. If the supremum is also the maximum, `Bool` is unified with the atom `true` and, otherwise, unified with the atom `false`.

`ppl_Polyhedron_minimize(+Handle, +LinExpr, ?Integer, ?Integer, ?Bool)` Succeeds if and only if the polyhedron P referenced by `Handle` is not empty and `LinExpr` is bounded from below in P.

`Integer1` is unified with the numerator of the infimum value and `Integer2` with the denominator of the infimum value. If the infimum is also the minimum, `Bool` is unified with the atom `true` and, otherwise, unified with the atom `false`.

`ppl_Polyhedron_minimize_with_point(+Handle, +LinExpr, ?Integer, ?Integer, ?Bool, ?Point)` Succeeds if and only if the polyhedron P referenced by `Handle` is not empty and `LinExpr` is bounded from below in P.

`Integer1` is unified with the numerator of the infimum value, `Integer2` with the denominator of the infimum value, and `Point` with a point or closure point where `LinExpr` reaches this value. If the infimum is also the minimum, `Bool` is unified with the atom `true` and, otherwise, unified with the atom `false`.

`ppl_Polyhedron_is_topologically_closed(+Handle)` Succeeds if and only if the polyhedron referenced by `Handle` is topologically closed.

`ppl_Polyhedron_contains_Polyhedron(+Handle_1, +Handle_2)` Succeeds if and only if the polyhedron referenced by `Handle_1` is included in or equal to the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_strictly_contains_Polyhedron(+Handle_1, +Handle_2)` Succeeds if and only if the polyhedron referenced by `Handle_1` is included in but not equal to the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_is_disjoint_from_Polyhedron(+Handle_1, +Handle_2)` Succeeds if and only if the polyhedron referenced by `Handle_1` is disjoint from the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_equals_Polyhedron(+Handle_1, +Handle_2)` Succeeds if and only if the polyhedron referenced by `Handle_1` is equal to the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_OK(+Handle)` Succeeds only if the polyhedron referenced by `Handle` is well formed, i.e., if it satisfies all its implementation invariants. Useful for debugging purposes.

`ppl_Polyhedron_add_constraint(+Handle, +Constraint)` `ppl_Polyhedron_add_constraint_and_minimize(+Handle, +Constraint)` Updates the polyhedron referenced by `Handle` to one obtained by adding `Constraint` to its constraint system. Thus, the query

```
?- ppl_new_Polyhedron_from_dimension(c, 3, X),
   A = '$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   ppl_Polyhedron_add_constraint(X, 4*A + B - 2*C >= 5).
```

will update the polyhedron with handle `X` to consist of the set of points in the vector space $\mathbb{R}^3$ satisfying the constraint $4x + y - 2z >= 5$.

Note that `ppl_Polyhedron_add_constraint_and_minimize/2` will fail if, after adding the constraint, the polyhedron is empty.

`ppl_Polyhedron_add_generator(+Handle, +Generator)`

`ppl_Polyhedron_add_generator_and_minimize(+Handle, +Generator)` Updates the polyhedron referenced by `Handle` to one obtained by adding `Generator` to its generator system. Thus, after the query

```
?- ppl_new_Polyhedron_from_dimension(c, 3, X),
   A = '$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   ppl_Polyhedron_add_generator(X, point(-100*A - 5*B, 8)).
```

will update the polyhedron with handle `X` to be the single point $(-12.5, -0.625, 0)^{\mathrm{T}}$ in the vector space $\mathbb{R}^3$.

`ppl_Polyhedron_add_constraints(+Handle, +Constraint_System)` Updates the polyhedron referenced by `Handle` to one obtained by adding to its constraint system the constraints in `Constraint_System`. E.g.,

```
| ?- ppl_new_Polyhedron_from_dimension(c, 2, X),
     A = '$VAR'(0), B = '$VAR'(1),
     ppl_Polyhedron_add_constraints(X, [4*A + B >= 3, A = 1]),
     ppl_Polyhedron_get_constraints(X, CS).

CS = [4*A+1*B>=3,1*A=1] ?
```

The updated polyhedron referenced by `Handle` can be empty and a query will succeed even when `Constraint_System` is unsatisfiable.

`ppl_Polyhedron_add_constraints_and_minimize(+Handle, +Constraint_System)` Updates the polyhedron referenced by `Handle` to one obtained by adding to its constraint system the constraints in `Constraint_System`. E.g.,

```
?- ppl_new_Polyhedron_from_dimension(c, 2, X),
   A = '$VAR'(0), B = '$VAR'(1),
   ppl_Polyhedron_add_constraints_and_minimize(X, [4*A + B >= 3, A = 1]),
   ppl_Polyhedron_get_constraints(X, CS).

CS = [1*B>= -1,1*A=1]
```

This will fail if, after adding the constraints, the polyhedron is empty. E.g., the following will fail,

```
?- A = '$VAR'(0), B = '$VAR'(1),
   ppl_new_Polyhedron_from_dimension(c, 2, X),
   ppl_Polyhedron_add_constraints_and_minimize(X,
     [4*A + B >= 3, A = 0, B =< 0]),
   ppl_Polyhedron_get_constraints(X, CS).
```

`ppl_Polyhedron_add_generators(+Handle, +Generator_System)` Updates the polyhedron referenced by `Handle` to one obtained by adding to its generator system the generators in `Generator_System`.

If the system of generators representing a polyhedron is non-empty, then it must include a point (see the paragraph on generator representation in Section **Representations of Convex Polyhedra**(p. 3)). Thus care must be taken to ensure that, before calling this predicate, either the polyhedron referenced by `Handle` is non-empty or that whenever `Generator_System` is non-empty the first element defines a point. E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 3, X),
   A='$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   ppl_Polyhedron_add_generators(X,
     [point(1*A + 1*B + 1*C, 1), ray(1*A), ray(2*A)]),
   ppl_Polyhedron_get_generators(X, GS).

GS = [ray(2*A), point(1*A+1*B+1*C), ray(1*A)]
```

`ppl_Polyhedron_add_generators_and_minimize(+Handle, +Generator_System)` Updates the polyhedron referenced by `Handle` to one obtained by adding to its generator system the generators in `Generator_System`.

Unlike the predicate `ppl_add_generators`, the order of the generators in `Generator_System` is not important. E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 3, X),
   A='$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   ppl_Polyhedron_add_generators_and_minimize(X,
     [ray(1*A), ray(2*A), point(1*A + 1*B + 1*C, 1)]),
   ppl_Polyhedron_get_generators(X, GS).

GS = [point(1*A+1*B+1*C), ray(1*A)]
```

`ppl_Polyhedron_intersection_assign(+Handle_1, +Handle_2)`

`ppl_Polyhedron_intersection_assign_and_minimize(+Handle_1, +Handle_2)` Assigns to the polyhedron referenced by `Handle_1` its intersection with the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_poly_hull_assign(+Handle_1, +Handle_2)`

`ppl_Polyhedron_poly_hull_assign_and_minimize(+Handle_1, +Handle_2)` Assigns to the polyhedron referenced by `Handle_1` its poly-hull with the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_poly_difference_assign(+Handle_1, +Handle_2)` Assigns to the polyhedron referenced by `Handle_1` its poly-difference with the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_affine_image(+Handle, +PPL_Var, +LinExpr, +Integer)`
Transforms the polyhedron referenced by `Handle` assigning the affine expression `LinExpr/Integer` to `PPL_Var`.

`ppl_Polyhedron_affine_preimage(+Handle, +PPL_Var, +LinExpr, +Integer)`
This is the inverse transformation to that for `ppl_affine_image`.

`ppl_Polyhedron_generalized_affine_image(+Handle, +PPL_Var, +Relation_-Symbol +LinExpr, +Integer)` Transforms the polyhedron referenced by `Handle` assigning the generalized affine image with respect to the transfer function `PPL_Var Relation_Symbol LinExpr/Integer`.

`ppl_Polyhedron_generalized_affine_image_lhs_rhs(+Handle, +LinExpr1, +Relation_Symbol +LinExpr2)` Transforms the polyhedron referenced by `Handle` assigning the generalized affine image with respect to the transfer function `LinExpr1 Relation_Symbol LinExpr2`.

`ppl_Polyhedron_time_elapse_assign(+Handle_1, +Handle_2)` Assigns to the polyhedron $\mathcal{P}$ referenced by `Handle_1` the time-elapse ($\mathcal{P} \nearrow \mathcal{Q}$) with the polyhedron $\mathcal{Q}$ referenced by `Handle_2`.

`ppl_Polyhedron_BHRZ03_widening_assign_with_token(+Handle_1, +Handle_2, ?Integer)` The polyhedra referenced by `Handle_1` and `Handle_2` are unaltered. The token `Integer` is 0 if a BHRZ03 widening would have changed the polyhedron referenced by `Handle_1` and is 1 otherwise.

`ppl_Polyhedron_BHRZ03_widening_assign(+Handle_1, +Handle_2)` Assigns to the polyhedron referenced by `Handle_1` its BHRZ03-widening with the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_limited_BHRZ03_extrapolation_assign_with_token( +Handle_1, +Handle_2, +Constraint_System, ?Integer)` The polyhedra referenced by `Handle_1` and `Handle_2` are unaltered. The token `Integer` is 0 if a BHRZ03-widening with the polyhedron referenced by `Handle_2`, improved by enforcing those constraints in `Constraint_-System` would have changed the polyhedron referenced by `Handle_1` and is 1 otherwise.

`ppl_Polyhedron_limited_BHRZ03_extrapolation_assign(+Handle_1, +Handle_2, +Constraint_System)` Assigns to the polyhedron $\mathcal{P}$ referenced by `Handle_1` the result of its BHRZ03-widening with the polyhedron referenced by `Handle_2`, improved by enforcing those constraints in `Constraint_System`.

`ppl_Polyhedron_bounded_BHRZ03_extrapolation_assign_with_token( +Handle_1, +Handle_2, +Constraint_System, ?Integer)` The polyhedra $\mathcal{P}_1$ and $\mathcal{P}_2$ referenced by `Handle_1` and `Handle_2`, respectively are unaltered. The token `Integer` is 0 if a BHRZ03-widening with $\mathcal{P}_2$, improved by enforcing all the constraints of the form $\pm x \leq r$ and $\pm x < r$ that are satisfied by all the points of $\mathcal{P}_1$ together with the constraints in `Constraint_System` would have changed the polyhedron referenced by `Handle_1` and is 1 otherwise.

`ppl_Polyhedron_bounded_BHRZ03_extrapolation_assign(+Handle_1, +Handle_2, +Constraint_System)` Assigns to the polyhedron $\mathcal{P}$ referenced by `Handle_1` the result of its BHRZ03-widening with the polyhedron referenced by `Handle_2` improved by enforcing all the constraints of the form $\pm x \leq r$ and $\pm x < r$ that are satisfied by all the points of $\mathcal{P}$ together with the constraints in `Constraint_System`.

`ppl_Polyhedron_H79_widening_assign_with_token(+Handle_1, +Handle_2, ?Integer)` The polyhedra referenced by `Handle_1` and `Handle_2` are unaltered. The token `Integer` is 0 if an H79 widening would have changed the polyhedron referenced by `Handle_1` and is 1 otherwise.

`ppl_Polyhedron_H79_widening_assign(+Handle_1, +Handle_2)` Assigns to the polyhedron referenced by `Handle_1` its H79-widening with the polyhedron referenced by `Handle_2`.

`ppl_Polyhedron_limited_H79_extrapolation_assign_with_token(+Handle_1, +Handle_2, +Constraint_System, ?Integer)` The polyhedra referenced by `Handle_1` and `Handle_2` are unaltered. The token `Integer` is 0 if a H79-widening with the polyhedron referenced by `Handle_2`, improved by enforcing those constraints in `Constraint_System` would have changed the polyhedron referenced by `Handle_1` and is 1 otherwise.

`ppl_Polyhedron_limited_H79_extrapolation_assign(+Handle_1, +Handle_2, +Constraint_System)` Assigns to the polyhedron $\mathcal{P}$ referenced by `Handle_1` its H79-widening with the polyhedron referenced by `Handle_2`, improved by enforcing those constraints in `Constraint_System`.

`ppl_Polyhedron_bounded_H79_extrapolation_assign_with_token(+Handle_1, +Handle_2, +Constraint_System, ?Integer)` The polyhedra $\mathcal{P}_1$ and $\mathcal{P}_2$ referenced by `Handle_1` and `Handle_2`, respectively are unaltered. The token `Integer` is 0 if a H79-widening with $\mathcal{P}_2$, improved by enforcing all the constraints of the form $\pm x \leq r$ and $\pm x < r$ that are satisfied by all the points of $\mathcal{P}_1$ together with the constraints in `Constraint_System` would have changed the polyhedron referenced by `Handle_1` and is 1 otherwise.

`ppl_Polyhedron_bounded_H79_extrapolation_assign(+Handle_1, +Handle_2, +Constraint_System)` Assigns to the polyhedron $\mathcal{P}$ referenced by `Handle_1` the result of its H79-widening with the polyhedron referenced by `Handle_2` improved by enforcing all the constraints of the form $\pm x \leq r$ and $\pm x < r$ that are satisfied by all the points of $\mathcal{P}$ together with the constraints in `Constraint_System`.

`ppl_Polyhedron_topological_closure_assign(+Handle)` Assigns to the polyhedron referenced by `Handle` its topological closure.

`ppl_Polyhedron_add_dimensions_and_embed(+Handle, +Integer)` Embeds the polyhedron referenced by `Handle` in a space that is enlarged by `Integer` dimensions, E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 0, X),
   ppl_Polyhedron_add_dimensions_and_embed(X, 2),
   ppl_Polyhedron_get_constraints(X, CS),
   ppl_Polyhedron_get_generators(X, GS).
```

```
CS = [],
GS = [point(0),line(1*A),line(1*B)]
```

`ppl_Polyhedron_concatenate_assign(+Handle1, +Handle2)` Updates the polyhedron $\mathcal{P}_1$ referenced by `Handle1` by first embedding $\mathcal{P}_1$ in a new space enlarged by the space dimensions of the polyhedron $\mathcal{P}_2$ referenced by `Handle2`, and then adds to its system of constraints a renamed-apart version of the constraints of $\mathcal{P}_2$.

E.g.,

```
?- ppl_new_Polyhedron_from_dimension(nnc, 2, X),
   A = '$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   D = '$VAR'(3), E = '$VAR'(4),
   ppl_new_Polyhedron_from_constraints(nnc, [A > 1, B >= 0, C >= 0], Y),
   ppl_Polyhedron_concatenate_assign(X, Y),
   ppl_Polyhedron_get_constraints(X, CS).

CS = [1*C > 1, 1*D >= 0, 1*E >= 0]
```

`ppl_Polyhedron_add_dimensions_and_project(+Handle, +Integer)` Projects the polyhedron referenced by `Handle` onto a space that is enlarged by `Integer` dimensions, E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 0, X),
   ppl_Polyhedron_add_dimensions_and_project(X, 2),
   ppl_Polyhedron_get_constraints(X, CS),
   ppl_Polyhedron_get_generators(X, GS).

CS = [1*A = 0, 1*B = 0],
GS = [point(0)]
```

`ppl_Polyhedron_remove_dimensions(+Handle, +List_of_PPL_Vars)` Removes the space dimensions given by the identifiers of the PPL variables in list `List_of_PPL_Vars` from the polyhedron referenced by `Handle`. The identifiers for the remaining PPL variables are renumbered so that they are consecutive and the maximum index is less than the number of dimensions. E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 3, X),
   A='$VAR'(0), B = '$VAR'(1), C = '$VAR'(2),
   ppl_Polyhedron_remove_dimensions(X, [B]),
   ppl_Polyhedron_space_dimension(X, K),
   ppl_Polyhedron_get_generators(X, GS).

K = 2,
GS = [point(0),line(1*A),line(1*B),line(0)]
```

`ppl_Polyhedron_remove_higher_dimensions(+Handle, +Integer))` Projects the polyhedron referenced to by `Handle` onto the first `Integer` dimension. E.g.,

```
?- ppl_new_Polyhedron_empty_from_dimension(c, 5, X),
   ppl_Polyhedron_remove_higher_dimensions(X, 3),
   ppl_Polyhedron_space_dimension(X, K).
```

`ppl_Polyhedron_expand_dimension(+Handle, +PPL_Var, +Integer))` `Integer` copies of the space dimension referenced by `PPL_Var` are added to the polyhedron referenced to by `Handle`.

`ppl_Polyhedron_fold_dimensions(+Handle, +List_of_PPL_Vars, +PPL_Var))`
The space dimensions referenced by the PPL variables in list `List_of_PPL_Vars` are folded into the dimension referenced by `PPL_Var` and removed. The result is undefined if `List_of_PPL_Vars` does not have the properties described in the paragraph **specifying the fold_dimensions operator**(p. 8) in Section **Operations on Convex Polyhedra**(p. 6).

`ppl_Polyhedron_map_dimensions(+Handle, +P_Func))` Maps the dimensions of the polyhedron referenced by `Handle` using the partial function defined by `P_Func`. The result is undefined if `P_Func` does not encode a partial function with the properties described in the paragraph **specifying the map_dimensions operator**(p. 7) in Section **Operations on Convex Polyhedra**(p. 6).

Compilation and Installation

When the Parma Polyhedra Library is configured, it tests for the existence of each supported Prolog system. If a supported Prolog system is correctly installed in a standard location, things are arranged so that the corresponding interface is built and installed.

In the sequel, `prefix` is the prefix under which you have installed the library (typically `/usr` or `/usr/local`).

As an option, the Prolog interface can track the creation and disposal of polyhedra. In fact, differently from native Prolog data, PPL polyhedra must be explicitly disposed and forgetting to do so is a very common mistake. To enable this option, configure the library adding `-DPROLOG_TRACK_ALLOCATION` to the options passed to the C++ compiler. Your configure command would then look like

```
path/to/configure --with-cxxflags="-DPROLOG_TRACK_ALLOCATION" ...
```

System-Dependent Features

CIAO Prolog Support for CIAO Prolog is under development and will be available in a future release. Only Ciao Prolog 1.9 #44 or later is supported.

GNU Prolog The GNU Prolog interface to the PPL library is available both as "PPL enhanced" GNU Prolog interpreter and as a library that can be linked to GNU Prolog programs. Only GNU Prolog version 1.2.12 or later is supported.

Notice that GNU Prolog version 1.2.12 suffers from a serious limitation as far as foreign code is concerned. In order to be safe you must configure GNU Prolog with the `-disable-ebp` option (note that this has a negative effect on performance). See `http://www.cs.unipr.it/pipermail/ppl-devel/2002-June/001777.html`, `http://www.cs.unipr.it/pipermail/ppl-devel/2002-June/001780.html`, `http://www.cs.unipr.it/pipermail/ppl-devel/2002-June/001788.html` and `http://www.cs.unipr.it/pipermail/ppl-devel/2002-June/001789.html` for more information.

We have experienced other serious problems with the GNU Prolog interface, up to and including GNU Prolog version 1.2.16: see `http://www.cs.unipr.it/pipermail/ppl-devel/2002-October/002657.html` for more information.

The `ppl_gprolog` **Executable** If an appropriate version of GNU Prolog is installed on the machine on which you compiled the library, the command `make install` will install the executable `ppl_gprolog` in the directory `prefix/bin`. The `ppl_gprolog` executable is simply the GNU Prolog interpreter with the Parma Polyhedra library linked in. The only thing you should do to use the library is to call `ppl_initialize/0` before any other PPL predicate and to call `ppl_finalize/0` when you are done with the library.

Linking the Library To GNU Prolog Programs In order to allow linking GNU Prolog programs to the PPL, the following files are installed in the directory `prefix/lib/ppl`: `ppl_gprolog.pl` contains the required foreign declarations; `libppl_gprolog.*` contain the executable code for the GNU Prolog interface in various formats (static library, shared library, libtool library). If your GNU Prolog program is constituted by, say, `source1.pl` and `source2.pl` and you want to create the executable `myprog`, your compilation command may look like

```
gplc -o myprog prefix/lib/ppl/ppl_gprolog.pl source1.pl source2.pl \
  -L '-Lprefix/lib/ppl -lppl_gprolog -Lprefix/lib -lppl -lgmpxx -lgmp -lstdc++'
```

SICStus Prolog The SICStus Prolog interface to the PPL library is available both as a statically linked module or as a dynamically linked one. Only SICStus Prolog version 3.9.0 or later is supported.

The Statically Linked `ppl_sicstus` **Executable** If an appropriate version of SICStus Prolog is installed on the machine on which you compiled the library, the command `make install` will install the executable `ppl_sicstus` in the directory `prefix/bin`. The `ppl_sicstus` executable is simply the SICStus Prolog system with the Parma Polyhedra library statically linked. The only thing you should do to use the library is to load `prefix/lib/ppl/ppl_sicstus.pl`.

Loading the SICStus Interface Dynamically In order to dynamically load the library from SICStus Prolog you should simply load `prefix/lib/ppl/ppl_sicstus.pl`. Notice that, for dynamic linking to work, you should have configured the library with the `-enable-shared` option.

SWI-Prolog The SWI-Prolog interface of the library is available both as a statically linked module or as a dynamically linked one. Only SWI-Prolog version 5.0 or later is supported.

The `ppl_pl` **Executable** If an appropriate version of SWI-Prolog is installed on the machine on which you compiled the library, the command `make install` will install the executable `ppl_pl` in the directory `prefix/bin`. The `ppl_pl` executable is simply the SWI-Prolog shell with the Parma Polyhedra library statically linked: from within `ppl_pl` all the services of the library are available without further action.

Loading the SWI-Prolog Interface Dynamically In order to dynamically load the library from SWI-Prolog you should simply load `prefix/lib/ppl/ppl_swiprolog.pl`. This will invoke `ppl_initialize/0` automatically but, at least for SWI-Prolog versions up to 5.0.7, it is the programmer's responsibility to call `ppl_finalize/0`. Alternatively, you can load the library directly with

```
:- load_foreign_library('prefix/lib/ppl/libppl_swiprolog').
```

This will call `ppl_initialize/0` automatically. Analogously,

```
:- unload_foreign_library('prefix/lib/ppl/libppl_swiprolog').
```

will, as part of the unload process, invoke `ppl_finalize/0`.

Notice that, for dynamic linking to work, you should have configured the library with the `-enable-shared` option.

XSB The XSB Prolog interface to the PPL library is available as a dynamically linked module. Only XSB version 2.5 and following is supported.

In order to dynamically load the library from XSB you should load the `ppl_xsb` module and import the predicates you need. For things to work, you may have to copy the files `prefix/lib/ppl/ppl_xsb.xwam` and `prefix/lib/ppl/ppl_xsb.so` in your current directory or in one of the XSB library directories.

YAP The YAP Prolog interface to the PPL library is available as a dynamically linked module. Only YAP version 4.4 or later is supported.

In order to dynamically load the library from YAP you should simply load `prefix/lib/ppl/ppl_yap.pl`. This will invoke `ppl_initialize/0` automatically; it is the programmer's responsibility to call `ppl_finalize/0` when the PPL library is no longer needed. Notice that, for dynamic linking to work, you should have configured the library with the `-enable-shared` option.

8 PPL Namespace Documentation

8.1 Parma_Polyhedra_Library Namespace Reference

The entire library is confined to this namespace.

Classes

- class **Parma_Polyhedra_Library::Variable**

  A dimension of the space.

- struct **Parma_Polyhedra_Library::Variable::Compare**

  Binary predicate defining the total ordering on variables.

- class **Parma_Polyhedra_Library::LinExpression**

  A linear expression.

- class **Parma_Polyhedra_Library::Constraint**

  A linear equality or inequality.

- class **Parma_Polyhedra_Library::Generator**

  A line, ray, point or closure point.

- class **Parma_Polyhedra_Library::Poly_Con_Relation**

  The relation between a polyhedron and a constraint.

- class **Parma_Polyhedra_Library::Poly_Gen_Relation**

  The relation between a polyhedron and a generator.

- class **Parma_Polyhedra_Library::BHRZ03_Certificate**

    The convergence certificate for the BHRZ03 widening operator.

- struct **Parma_Polyhedra_Library::BHRZ03_Certificate::Compare**

    A total ordering on BHRZ03 certificates.

- class **Parma_Polyhedra_Library::H79_Certificate**

    A convergence certificate for the H79 widening operator.

- struct **Parma_Polyhedra_Library::H79_Certificate::Compare**

    A total ordering on H79 certificates.

- class **Parma_Polyhedra_Library::Polyhedron**

    The base class for convex polyhedra.

- class **Parma_Polyhedra_Library::C_Polyhedron**

    A closed convex polyhedron.

- class **Parma_Polyhedra_Library::NNC_Polyhedron**

    A not necessarily closed convex polyhedron.

- class **Parma_Polyhedra_Library::Determinate**< **PH** >

    Wraps a PPL class into a determinate constraint system interface.

- class **Parma_Polyhedra_Library::PowerSet**< **CS** >

    The powerset construction on constraint systems.

- class **Parma_Polyhedra_Library::Polyhedra_PowerSet**< **PH** >

    The powerset construction instantiated on PPL polyhedra.

Typedefs

- typedef mpz_class **Integer**

    See the GMP's manual available at `http://swox.com/gmp/`.

- typedef std::set< **Variable**, **Variable::Compare** > **Variables_Set**

    An std::set containing variables in increasing order of dimension index.

Functions

- unsigned **version_major** ()

    Returns the major number of the PPL version.

- unsigned **version_minor** ()

    Returns the minor number of the PPL version.

- unsigned **version_revision** ()

Returns the revision number of the PPL version.

- unsigned **version_beta** ()

  Returns the beta number of the PPL version.

- const char ∗ **version** ()

  Returns a character string containing the PPL version.

- const char ∗ **banner** ()

  Returns a character string containing the PPL banner.

8.1.1 Detailed Description

The entire library is confined to this namespace.

8.1.2 Function Documentation

8.1.2.1 const char∗ banner ()

Returns a character string containing the PPL banner.

The banner provides information about the PPL version, the licensing, the lack of any warranty whatsoever, the C++ compiler used to build the library, where to report bugs and where to look for further information.

8.2 Parma_Polyhedra_Library::IO_Operators Namespace Reference

All input/output operators are confined to this namespace.

8.2.1 Detailed Description

All input/output operators are confined to this namespace.

This is done so that the library's input/output operators do not interfere with those the user might want to define. In fact, it is highly unlikely that any pre-defined I/O operator will suit the needs of a client application. On the other hand, those applications for which the PPL I/O operator are enough can easily obtain access to them. For example, a directive like

```
using namespace Parma_Polyhedra_Library::IO_Operators;
```

would suffice for most uses. In more complex situations, such as

```
const ConSys& cs = ...;
copy(cs.begin(), cs.end(),
     ostream_iterator<Constraint>(cout, "\n"));
```

the **Parma_Polyhedra_Library**(p. 67) namespace must be suitably extended. This can be done as follows:

```
namespace Parma_Polyhedra_Library {
  // Import all the output operators into the main PPL namespace.
  using IO_Operators::operator<<;
}
```

8.3 std Namespace Reference

The standard C++ namespace.

8.3.1 Detailed Description

The standard C++ namespace.

The Parma Polyhedra Library conforms to the C++ standard and, in particular, as far as reserved names are concerned (17.4.3.1, [lib.reserved.names]). The PPL, however, defines several template specializations for the standard library templates swap() and iter_swap() (25.2.2, [lib.alg.swap]).

9 PPL Class Documentation

9.1 Parma_Polyhedra_Library::BHRZ03_Certificate Class Reference

The convergence certificate for the BHRZ03 widening operator.

Public Member Functions

- **BHRZ03_Certificate** ()

  Default constructor.

- **BHRZ03_Certificate** (const **Polyhedron** &ph)

  Constructor: computes the certificate for `ph`.

- **BHRZ03_Certificate** (const **BHRZ03_Certificate** &y)

  Copy constructor.

- ~**BHRZ03_Certificate** ()

  Destructor.

- int **compare** (const **BHRZ03_Certificate** &y) const

  The comparison function for certificates.

- int **compare** (const **Polyhedron** &ph) const

  Compares `*this` *with the certificate for polyhedron* `ph`.

9.1.1 Detailed Description

The convergence certificate for the BHRZ03 widening operator.

Convergence certificates are used to instantiate the BHZ03 framework so as to define widening operators for the finite powerset domain.

Note:

Each convergence certificate has to be used together with a compatible widening operator. In particular, **BHRZ03_Certificate**(p. 70) can certify the convergence of both the BHRZ03 and the H79 widenings.

9.1.2 Member Function Documentation

9.1.2.1 int Parma_Polyhedra_Library::BHRZ03_Certificate::compare (const BHRZ03_Certificate & *y*) const

The comparison function for certificates.

Returns:

-1, 0 or 1 depending on whether `*this` is smaller than, equal to, or greater than `y`, respectively.

Compares `*this` with `y`, using a total ordering which is a refinement of the limited growth ordering relation for the BHRZ03 widening.

9.2 Parma_Polyhedra_Library::BHRZ03_Certificate::Compare Struct Reference

A total ordering on BHRZ03 certificates.

Public Member Functions

- bool **operator()** (const **BHRZ03_Certificate** &x, const **BHRZ03_Certificate** &y) const
  Returns `true` *if and only if* `x` *comes before* `y`.

9.2.1 Detailed Description

A total ordering on BHRZ03 certificates.

This binary predicate defines a total ordering on BHRZ03 certificates which is used when storing information about sets of polyhedra.

9.3 Parma_Polyhedra_Library::C_Polyhedron Class Reference

A closed convex polyhedron.

Inherits **Parma_Polyhedra_Library::Polyhedron**.

Public Member Functions

- **C_Polyhedron** (dimension_type num_dimensions=0, **Degenerate_Kind** kind=UNIVERSE)
  Builds either the universe or the empty C polyhedron.
- **C_Polyhedron** (const ConSys &cs)
  Builds a C polyhedron from a system of constraints.
- **C_Polyhedron** (ConSys &cs)
  Builds a C polyhedron recycling a system of constraints.
- **C_Polyhedron** (const GenSys &gs)
  Builds a C polyhedron from a system of generators.

- **C_Polyhedron** (GenSys &gs)

  Builds a C polyhedron recycling a system of generators.

- **C_Polyhedron** (const **NNC_Polyhedron** &y)

  Builds a C polyhedron from the NNC polyhedron `y`.

- template<typename Box> **C_Polyhedron** (const Box &box, From_Bounding_Box dummy)

  Builds a C polyhedron out of a generic, interval-based bounding box.

- **C_Polyhedron** (const **C_Polyhedron** &y)

  Ordinary copy-constructor.

- **C_Polyhedron** & **operator=** (const **C_Polyhedron** &y)

  The assignment operator. (`*this` *and* `y` *can be dimension-incompatible.).*

- ~**C_Polyhedron** ()

  Destructor.

Static Public Member Functions

- dimension_type **max_space_dimension** ()

  Returns the maximum space dimension a **C_Polyhedron**(p. 71) *can handle.*

9.3.1 Detailed Description

A closed convex polyhedron.

An object of the class **C_Polyhedron**(p. 71) represents a *topologically closed* convex polyhedron in the vector space $\mathbb{R}^n$.

When building a closed polyhedron starting from a system of constraints, an exception is thrown if the system contains a *strict inequality* constraint. Similarly, an exception is thrown when building a closed polyhedron starting from a system of generators containing a *closure point*.

Note:

Such an exception will be obtained even if the system of constraints (resp., generators) actually defines a topologically closed subset of the vector space, i.e., even if all the strict inequalities (resp., closure points) in the system happen to be redundant with respect to the system obtained by removing all the strict inequality constraints (resp., all the closure points). In contrast, when building a closed polyhedron starting from an object of the class **NNC_Polyhedron**(p. 93), the precise topological closure test will be performed.

9.3.2 Constructor & Destructor Documentation

9.3.2.1 Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (dimension_type *num_dimensions* = 0, Degenerate_Kind *kind* = UNIVERSE) [explicit]

Builds either the universe or the empty C polyhedron.

Parameters:

num_dimensions The number of dimensions of the vector space enclosing the C polyhedron;

kind Specifies whether a universe or an empty C polyhedron should be built.

Exceptions:

std::length_error Thrown if `num_dimensions` exceeds the maximum allowed space dimension.

Both parameters are optional: by default, a 0-dimension space universe C polyhedron is built.

9.3.2.2 Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (const ConSys & *cs*)

Builds a C polyhedron from a system of constraints.

The polyhedron inherits the space dimension of the constraint system.

Parameters:

cs The system of constraints defining the polyhedron.

Exceptions:

std::invalid_argument Thrown if the system of constraints contains strict inequalities.

9.3.2.3 Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (ConSys & *cs*)

Builds a C polyhedron recycling a system of constraints.

The polyhedron inherits the space dimension of the constraint system.

Parameters:

cs The system of constraints defining the polyhedron. It is not declared `const` because its data-structures will be recycled to build the polyhedron.

Exceptions:

std::invalid_argument Thrown if the system of constraints contains strict inequalities.

9.3.2.4 Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (const GenSys & *gs*)

Builds a C polyhedron from a system of generators.

The polyhedron inherits the space dimension of the generator system.

Parameters:

gs The system of generators defining the polyhedron.

Exceptions:

std::invalid_argument Thrown if the system of generators is not empty but has no points, or if it contains closure points.

9.3.2.5 Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (GenSys & *gs*)

Builds a C polyhedron recycling a system of generators.

The polyhedron inherits the space dimension of the generator system.

Parameters:

gs The system of generators defining the polyhedron. It is not declared `const` because its data-structures will be recycled to build the polyhedron.

Exceptions:

std::invalid_argument Thrown if the system of generators is not empty but has no points, or if it contains closure points.

9.3.2.6 Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (const NNC_Polyhedron & *y*) `[explicit]`

Builds a C polyhedron from the NNC polyhedron `y`.

Exceptions:

std::invalid_argument Thrown if the polyhedron `y` is not topologically closed.

9.3.2.7 template<typename Box> Parma_Polyhedra_Library::C_Polyhedron::C_Polyhedron (const Box & *box*, From_Bounding_Box *dummy*)

Builds a C polyhedron out of a generic, interval-based bounding box.

For a description of the methods that should be provided by the template class Box, see the documentation of the protected method: template <typename Box> **Polyhedron::Polyhedron(Topology topol, const Box& box)**(p. 114);

Parameters:

box The bounding box representing the polyhedron to be built;

dummy A dummy tag to syntactically differentiate this one from the other constructors.

Exceptions:

std::invalid_argument Thrown if `box` has intervals that are not topologically closed (i.e., having some finite but open bounds).

9.4 Parma_Polyhedra_Library::Constraint Class Reference

A linear equality or inequality.

Public Types

- enum **Type** { **EQUALITY**, **NONSTRICT_INEQUALITY**, **STRICT_INEQUALITY** }

  The constraint type.

Public Member Functions

- **Constraint** (const **Constraint** &c)

  Ordinary copy-constructor.

- ∼**Constraint** ()

  Destructor.

- **Constraint** & **operator=** (const **Constraint** &c)

  Assignment operator.

- dimension_type **space_dimension** () const

  Returns the dimension of the vector space enclosing `*this`.

- **Type** **type** () const

  Returns the constraint type of `*this`.

- bool **is_equality** () const

  Returns `true` *if and only if* `*this` *is an equality constraint.*

- bool **is_inequality** () const

  Returns `true` *if and only if* `*this` *is an inequality constraint (either strict or non-strict).*

- bool **is_nonstrict_inequality** () const

  Returns `true` *if and only if* `*this` *is a non-strict inequality constraint.*

- bool **is_strict_inequality** () const

  Returns `true` *if and only if* `*this` *is a strict inequality constraint.*

- const **Integer** & **coefficient** (**Variable** v) const

  Returns the coefficient of `v` *in* `*this`.

- const **Integer** & **inhomogeneous_term** () const

  Returns the inhomogeneous term of `*this`.

- bool **OK** () const

  Checks if all the invariants are satisfied.

Static Public Member Functions

- const **Constraint** & **zero_dim_false** ()

  The unsatisfiable (zero-dimension space) constraint $0 = 1$.

- const **Constraint** & **zero_dim_positivity** ()

  The true (zero-dimension space) constraint $0 \leq 1$*, also known as* positivity constraint.

Related Functions

(Note that these are not member functions.)

- std::ostream & **operator**<< (std::ostream &s, const **Constraint** &c)

  Output operator.

- **Constraint operator==** (const **LinExpression** &e1, const **LinExpression** &e2)

  Returns the constraint `e1 = e2`.

- **Constraint operator==** (const **LinExpression** &e, const **Integer** &n)

  Returns the constraint `e = n`.

- **Constraint operator==** (const **Integer** &n, const **LinExpression** &e)

  Returns the constraint `n = e`.

- **Constraint operator<=** (const **LinExpression** &e1, const **LinExpression** &e2)

  Returns the constraint `e1 <= e2`.

- **Constraint operator<=** (const **LinExpression** &e, const **Integer** &n)

  Returns the constraint `e <= n`.

- **Constraint operator<=** (const **Integer** &n, const **LinExpression** &e)

  Returns the constraint `n <= e`.

- **Constraint operator>=** (const **LinExpression** &e1, const **LinExpression** &e2)

  Returns the constraint `e1 >= e2`.

- **Constraint operator>=** (const **LinExpression** &e, const **Integer** &n)

  Returns the constraint `e >= n`.

- **Constraint operator>=** (const **Integer** &n, const **LinExpression** &e)

  Returns the constraint `n >= e`.

- **Constraint operator<** (const **LinExpression** &e1, const **LinExpression** &e2)

  Returns the constraint `e1 < e2`.

- **Constraint operator<** (const **LinExpression** &e, const **Integer** &n)

  Returns the constraint `e < n`.

- **Constraint operator<** (const **Integer** &n, const **LinExpression** &e)

  Returns the constraint `n < e`.

- **Constraint operator>** (const **LinExpression** &e1, const **LinExpression** &e2)

  Returns the constraint `e1 > e2`.

- **Constraint operator>** (const **LinExpression** &e, const **Integer** &n)

  Returns the constraint `e > n`.

- **Constraint operator>** (const **Integer** &n, const **LinExpression** &e)

Returns the constraint n > e.

- void **swap** (**Parma_Polyhedra_Library::Constraint** &x, **Parma_Polyhedra_Library::Constraint** &y)

  Specializes `std::swap`.

9.4.1 Detailed Description

A linear equality or inequality.

An object of the class **Constraint**(p. 74) is either:

- an equality: $\sum_{i=0}^{n-1} a_i x_i + b = 0$;
- a non-strict inequality: $\sum_{i=0}^{n-1} a_i x_i + b \geq 0$; or
- a strict inequality: $\sum_{i=0}^{n-1} a_i x_i + b > 0$;

where n is the dimension of the space, a_i is the integer coefficient of variable x_i and b is the integer inhomogeneous term.

How to build a constraint

Constraints are typically built by applying a relation symbol to a pair of linear expressions. Available relation symbols are equality (==), non-strict inequalities (>= and <=) and strict inequalities (< and >). The space-dimension of a constraint is defined as the maximum space-dimension of the arguments of its constructor.

In the following examples it is assumed that variables x, y and z are defined as follows:

```
Variable x(0);
Variable y(1);
Variable z(2);
```

Example 1

The following code builds the equality constraint $3x + 5y - z = 0$, having space-dimension 3:

```
Constraint eq_c(3*x + 5*y - z == 0);
```

The following code builds the (non-strict) inequality constraint $4x \geq 2y - 13$, having space-dimension 2:

```
Constraint ineq_c(4*x >= 2*y - 13);
```

The corresponding strict inequality constraint $4x > 2y - 13$ is obtained as follows:

```
Constraint strict_ineq_c(4*x > 2*y - 13);
```

An unsatisfiable constraint on the zero-dimension space $\mathbb{R}^0$ can be specified as follows:

```
Constraint false_c = Constraint::zero_dim_false();
```

Equivalent, but more involved ways are the following:

```
Constraint false_c1(LinExpression::zero() == 1);
Constraint false_c2(LinExpression::zero() >= 1);
Constraint false_c3(LinExpression::zero() > 0);
```

In contrast, the following code defines an unsatisfiable constraint having space-dimension 3:

```
Constraint false_c(0*z == 1);
```

How to inspect a constraint

Several methods are provided to examine a constraint and extract all the encoded information: its space-dimension, its type (equality, non-strict inequality, strict inequality) and the value of its integer coefficients.

Example 2

The following code shows how it is possible to access each single coefficient of a constraint. Given an inequality constraint (in this case $x - 5y + 3z <= 4$), we construct a new constraint corresponding to its complement (thus, in this case we want to obtain the strict inequality constraint $x - 5y + 3z > 4$).

```
Constraint c1(x - 5*y + 3*z <= 4);
cout << "Constraint c1: " << c1 << endl;
if (c1.is_equality())
  cout << "Constraint c1 is not an inequality." << endl;
else {
  LinExpression e;
  for (int i = c1.space_dimension() - 1; i >= 0; i--)
    e += c1.coefficient(Variable(i)) * Variable(i);
  e += c1.inhomogeneous_term();
  Constraint c2 = c1.is_strict_inequality() ? (e <= 0) : (e < 0);
  cout << "Complement c2: " << c2 << endl;
}
```

The actual output is the following:

```
Constraint c1: -A + 5*B - 3*C >= -4
Complement c2: A - 5*B + 3*C > 4
```

Note that, in general, the particular output obtained can be syntactically different from the (semantically equivalent) constraint considered.

9.4.2 Member Enumeration Documentation

9.4.2.1 enum Parma_Polyhedra_Library::Constraint::Type

The constraint type.

Enumeration values:

EQUALITY The constraint is an equality.

NONSTRICT_INEQUALITY The constraint is a non-strict inequality.

STRICT_INEQUALITY The constraint is a strict inequality.

9.4.3 Member Function Documentation

9.4.3.1 const Integer& Parma_Polyhedra_Library::Constraint::coefficient (Variable *v*) const

Returns the coefficient of `v` in `*this`.

Exceptions:

std::invalid_argument thrown if the index of `v` is greater than or equal to the space-dimension of `*this`.

9.5 Parma_Polyhedra_Library::Determinate< PH > Class Template Reference

Wraps a PPL class into a determinate constraint system interface.

Public Member Functions

Constructors and Destructor

- **Determinate** (dimension_type num_dimensions=0, bool universe=true)
  Builds either the top or the bottom of the determinate constraint system defined on the vector space having `num_dimensions` *dimensions.*
- **Determinate** (const PH &p)
  Injection operator: builds the determinate constraint system element corresponding to the base-level element `p`.
- **Determinate** (const ConSys &cs)
  Injection operator: builds the determinate constraint system element corresponding to the base-level element represented by `cs`.
- **Determinate** (const **Determinate** &y)
  Copy constructor.
- ~**Determinate** ()
  Destructor.

Member Functions that Do Not Modify the Domain Element

- dimension_type **space_dimension** () const
  Returns the dimension of the vector space enclosing `*this`.
- const ConSys & **constraints** () const
  Returns the system of constraints.
- const ConSys & **minimized_constraints** () const
  Returns the system of constraints, with no redundant constraint.
- const PH & **element** () const
  Returns a const reference to the embedded element.
- PH & **element** ()
  Returns a reference to the embedded element.
- bool **is_top** () const
  Returns `true` *if and only if* `*this` *is the top of the determinate constraint system (i.e., the whole vector space).*
- bool **is_bottom** () const
  Returns `true` *if and only if* `*this` *is the bottom of the determinate constraint system (i.e., the emptyset).*
- bool **definitely_entails** (const **Determinate** &y) const
  Returns `true` *if and only if* `*this` *entails* `y` *(i.e.,* `*this` *is contained in* `y`*).*

- bool **is_definitely_equivalent_to** (const **Determinate** &y) const
  Returns `true` *if and only if* `*this` *and* `y` *are equivalent.*

- bool **OK** () const
  Checks if all the invariants are satisfied.

Space-Dimension Preserving Member Functions that May Modify the Domain Element

- void **upper_bound_assign** (const **Determinate** &y)
  Assigns to `*this` *the upper bound of* `*this` *and* `y`.

- void **meet_assign** (const **Determinate** &y)
  Assigns to `*this` *the meet of* `*this` *and* `y`.

- void **add_constraint** (const **Constraint** &c)
  Assigns to `*this` *the meet of* `*this` *and the element represented by constraint* `c`.

- void **add_constraints** (ConSys &cs)
  Assigns to `*this` *the meet of* `*this` *and the element represented by the constraints in* `cs`.

Member Functions that May Modify the Dimension of the Vector Space

- **Determinate** & **operator=** (const **Determinate** &y)
  Assignment operator.

- void **swap** (**Determinate** &y)
  Swaps `*this` *with* `y`.

- void **add_dimensions_and_embed** (dimension_type m)
  Adds `m` *new dimensions and embeds the old domain element in the new vector space.*

- void **add_dimensions_and_project** (dimension_type m)
  Adds `m` *new dimensions to the domain element and does not embed it in the new vector space.*

- void **concatenate_assign** (const **Determinate** &y)
  Assigns to `*this` *the* **concatenation**(p. 6) *of* `*this` *and* `y`*, taken in this order.*

- void **remove_dimensions** (const **Variables_Set** &to_be_removed)
  Removes all the specified dimensions.

- void **remove_higher_dimensions** (dimension_type new_dimension)
  Removes the higher dimensions so that the resulting space will have dimension `new_dimension`.

- template<typename PartialFunction> void **map_dimensions** (const PartialFunction &pfunc)
  Remaps the dimensions of the vector space according to a partial function.

Friends

- bool **operator==** (const **Determinate**< PH > &x, const **Determinate**< PH > &y)

  Returns `true` *if and only if* `x` *and* `y` *are the same domain element.*

- bool **operator!=** (const **Determinate**< PH > &x, const **Determinate**< PH > &y)

  Returns `true` *if and only if* `x` *and* `y` *are different domain elements.*

Related Functions

(Note that these are not member functions.)

- std::ostream & **operator**<< (std::ostream &, const **Determinate**< PH > &)

  Output operator.

- void **swap** (**Parma_Polyhedra_Library::Determinate**< PH > &x, **Parma_Polyhedra_Library::Determinate**< PH > &y)

  Specializes `std::swap`.

9.5.1 Detailed Description

template<typename PH> class Parma_Polyhedra_Library::Determinate< PH >

Wraps a PPL class into a determinate constraint system interface.

9.5.2 Constructor & Destructor Documentation

9.5.2.1 template<typename PH> Parma_Polyhedra_Library::Determinate< PH >::Determinate (dimension_type *num_dimensions* = `0`, bool *universe* = `true`) `[explicit]`

Builds either the top or the bottom of the determinate constraint system defined on the vector space having `num_dimensions` dimensions.

The top element, corresponding to the whole vector space, is built if `universe` is `true`; otherwise the bottom element, corresponding to the emptyset, is built. By default, the top of a zero-dimension vector space is built.

9.5.3 Member Function Documentation

9.5.3.1 template<typename PH> void Parma_Polyhedra_Library::Determinate< PH >::add_constraint (const Constraint & *c*)

Assigns to `*this` the meet of `*this` and the element represented by constraint `c`.

Exceptions:

std::invalid_argument Thrown if `*this` and constraint `c` are topology-incompatible or dimension-incompatible.

9.5.3.2 template<typename PH> void Parma_Polyhedra_Library::Determinate< PH >::add_-constraints (ConSys & *cs*)

Assigns to `*this` the meet of `*this` and the element represented by the constraints in `cs`.

Parameters:

cs The constraints to intersect with. This parameter is not declared `const` because it can be modified.

Exceptions:

std::invalid_argument Thrown if `*this` and `cs` are topology-incompatible or dimension-incompatible.

9.5.3.3 template<typename PH> void Parma_Polyhedra_Library::Determinate< PH >::remove_dimensions (const Variables_Set & *to_be_removed*)

Removes all the specified dimensions.

Parameters:

to_be_removed The set of **Variable**(p. 132) objects corresponding to the dimensions to be removed.

Exceptions:

std::invalid_argument Thrown if `*this` is dimension-incompatible with one of the **Variable**(p. 132) objects contained in `to_be_removed`.

9.5.3.4 template<typename PH> void Parma_Polyhedra_Library::Determinate< PH >::remove_higher_dimensions (dimension_type *new_dimension*)

Removes the higher dimensions so that the resulting space will have dimension `new_dimension`.

Exceptions:

std::invalid_argument Thrown if `new_dimensions` is greater than the space dimension of `*this`.

9.5.3.5 template<typename PH> template<typename PartialFunction> void Parma_Polyhedra_-Library::Determinate< PH >::map_dimensions (const PartialFunction & *pfunc*)

Remaps the dimensions of the vector space according to a partial function.

See **Polyhedron::map_dimensions**(p. 128).

9.5.4 Friends And Related Function Documentation

9.5.4.1 template<typename PH> bool operator== (const Determinate< PH > & *x*, const Determinate< PH > & *y*) `[friend]`

Returns `true` if and only if `x` and `y` are the same domain element.

Exceptions:

std::invalid_argument Thrown if `x` and `y` are topology-incompatible or dimension-incompatible.

9.5.4.2 template<typename PH> bool operator!= (const Determinate< PH > & *x*, const Determinate< PH > & *y*) [friend]

Returns `true` if and only if `x` and `y` are different domain elements.

Exceptions:

std::invalid_argument Thrown if `x` and `y` are topology-incompatible or dimension-incompatible.

9.6 Parma_Polyhedra_Library::Generator Class Reference

A line, ray, point or closure point.

Public Types

- enum **Type** { **LINE**, **RAY**, **POINT**, **CLOSURE_POINT** }

  The generator type.

Public Member Functions

- **Generator** (const **Generator** &g)

  Ordinary copy-constructor.

- ~**Generator** ()

  Destructor.

- **Generator** & **operator=** (const **Generator** &g)

  Assignment operator.

- dimension_type **space_dimension** () const

  Returns the dimension of the vector space enclosing `*this`.

- **Type type** () const

  Returns the generator type of `*this`.

- bool **is_line** () const

  Returns `true` *if and only if* `*this` *is a line.*

- bool **is_ray** () const

  Returns `true` *if and only if* `*this` *is a ray.*

- bool **is_point** () const

  Returns `true` *if and only if* `*this` *is a point.*

- bool **is_closure_point** () const

  Returns `true` *if and only if* `*this` *is a closure point.*

- const **Integer** & **coefficient** (**Variable** v) const

  Returns the coefficient of `v` *in* `*this`.

- const **Integer** & **divisor** () const

  If `*this` *is either a point or a closure point, returns its divisor.*

- bool **OK** () const

  Checks if all the invariants are satisfied.

Static Public Member Functions

- **Generator line** (const **LinExpression** &e)

  Shorthand for **Generator**(p. 83) **Generator::line(const LinExpression& e)**(p. 87).

- **Generator ray** (const **LinExpression** &e)

  Shorthand for **Generator**(p. 83) **Generator::ray(const LinExpression& e)**(p. 87).

- **Generator point** (const **LinExpression** &e=LinExpression::zero(), const **Integer** &d=Integer_-one())

  Shorthand for **Generator**(p. 83) **Generator::point(const LinExpression& e, const Integer& d)**(p. 87).

- **Generator closure_point** (const **LinExpression** &e=LinExpression::zero(), const **Integer** &d=Integer_one())

  Shorthand for **Generator**(p. 83) **Generator::closure_point(const LinExpression& e, const Integer& d)**(p. 88).

- const **Generator** & **zero_dim_point** ()

  Returns the origin of the zero-dimensional space $\mathbb{R}^0$.

- const **Generator** & **zero_dim_closure_point** ()

  Returns, as a closure point, the origin of the zero-dimensional space $\mathbb{R}^0$.

Related Functions

(Note that these are not member functions.)

- std::ostream & **operator**<< (std::ostream &s, const **Generator** &g)

  Output operator.

- void **swap** (**Parma_Polyhedra_Library::Generator** &x, **Parma_Polyhedra_-Library::Generator** &y)

  Specializes `std::swap`.

9.6.1 Detailed Description

A line, ray, point or closure point.

An object of the class **Generator**(p. 83) is one of the following:

- a line $\boldsymbol{l} = (a_0, \ldots, a_{n-1})^\mathrm{T}$;

- a ray $\boldsymbol{r} = (a_0, \ldots, a_{n-1})^{\mathrm{T}}$;
- a point $\boldsymbol{p} = (\frac{a_0}{d}, \ldots, \frac{a_{n-1}}{d})^{\mathrm{T}}$;
- a closure point $\boldsymbol{c} = (\frac{a_0}{d}, \ldots, \frac{a_{n-1}}{d})^{\mathrm{T}}$;

where n is the dimension of the space and, for points and closure points, $d > 0$ is the divisor.

A note on terminology.

As observed in Section **Representations of Convex Polyhedra**(p. 3), there are cases when, in order to represent a polyhedron $\mathcal{P}$ using the generator system $\mathcal{G} = (L, R, P, C)$, we need to include in the finite set P even points of $\mathcal{P}$ that are *not* vertices of $\mathcal{P}$. This situation is even more frequent when working with NNC polyhedra and it is the reason why we prefer to use the word 'point' where other libraries use the word 'vertex'.

How to build a generator.

Each type of generator is built by applying the corresponding function (`line`, `ray`, `point` or `closure_point`) to a linear expression, representing a direction in the space; the space-dimension of the generator is defined as the space-dimension of the corresponding linear expression. Linear expressions used to define a generator should be homogeneous (any constant term will be simply ignored). When defining points and closure points, an optional Integer argument can be used as a common *divisor* for all the coefficients occurring in the provided linear expression; the default value for this argument is 1.

In all the following examples it is assumed that variables `x`, `y` and `z` are defined as follows:

```
Variable x(0);
Variable y(1);
Variable z(2);
```

Example 1

The following code builds a line with direction $x - y - z$ and having space-dimension 3:

```
Generator l = line(x - y - z);
```

As mentioned above, the constant term of the linear expression is not relevant. Thus, the following code has the same effect:

```
Generator l = line(x - y - z + 15);
```

By definition, the origin of the space is not a line, so that the following code throws an exception:

```
Generator l = line(0*x);
```

Example 2

The following code builds a ray with the same direction as the line in Example 1:

```
Generator r = ray(x - y - z);
```

As is the case for lines, when specifying a ray the constant term of the linear expression is not relevant; also, an exception is thrown when trying to build a ray from the origin of the space.

Example 3

The following code builds the point $\boldsymbol{p} = (1, 0, 2)^{\mathrm{T}} \in \mathbb{R}^3$:

```
Generator p = point(1*x + 0*y + 2*z);
```

The same effect can be obtained by using the following code:

```
Generator p = point(x + 2*z);
```

Similarly, the origin $\mathbf{0} \in \mathbb{R}^3$ can be defined using either one of the following lines of code:

```
Generator origin3 = point(0*x + 0*y + 0*z);
Generator origin3_alt = point(0*z);
```

Note however that the following code would have defined a different point, namely $\mathbf{0} \in \mathbb{R}^2$:

```
Generator origin2 = point(0*y);
```

The following two lines of code both define the only point having space-dimension zero, namely $\mathbf{0} \in \mathbb{R}^0$. In the second case we exploit the fact that the first argument of the function `point` is optional.

```
Generator origin0 = Generator::zero_dim_point();
Generator origin0_alt = point();
```

Example 4

The point $\boldsymbol{p}$ specified in Example 3 above can also be obtained with the following code, where we provide a non-default value for the second argument of the function `point` (the divisor):

```
Generator p = point(2*x + 0*y + 4*z, 2);
```

Obviously, the divisor can be usefully exploited to specify points having some non-integer (but rational) coordinates. For instance, the point $\boldsymbol{q} = (-1.5, 3.2, 2.1)^{\mathrm{T}} \in \mathbb{R}^3$ can be specified by the following code:

```
Generator q = point(-15*x + 32*y + 21*z, 10);
```

If a zero divisor is provided, an exception is thrown.

Example 5

Closures points are specified in the same way we defined points, but invoking their specific constructor function. For instance, the closure point $\boldsymbol{c} = (1, 0, 2)^{\mathrm{T}} \in \mathbb{R}^3$ is defined by

```
Generator c = closure_point(1*x + 0*y + 2*z);
```

For the particular case of the (only) closure point having space-dimension zero, we can use any of the following:

```
Generator closure_origin0 = Generator::zero_dim_closure_point();
Generator closure_origin0_alt = closure_point();
```

How to inspect a generator

Several methods are provided to examine a generator and extract all the encoded information: its space-dimension, its type and the value of its integer coefficients.

Example 6

The following code shows how it is possible to access each single coefficient of a generator. If `g1` is a point having coordinates $(a_0, \ldots, a_{n-1})^{\mathrm{T}}$, we construct the closure point `g2` having coordinates $(a_0, 2a_1, \ldots, (i+1)a_i, \ldots, na_{n-1})^{\mathrm{T}}$.

```
if (g1.is_point()) {
  cout << "Point g1: " << g1 << endl;
  LinExpression e;
  for (int i = g1.space_dimension() - 1; i >= 0; i--)
    e += (i + 1) * g1.coefficient(Variable(i)) * Variable(i);
  Generator g2 = closure_point(e, g1.divisor());
  cout << "Closure point g2: " << g2 << endl;
}
else
  cout << "Generator g1 is not a point." << endl;
```

Therefore, for the point

```
Generator g1 = point(2*x - y + 3*z, 2);
```

we would obtain the following output:

```
Point g1: p((2*A - B + 3*C)/2)
Closure point g2: cp((2*A - 2*B + 9*C)/2)
```

When working with (closure) points, be careful not to confuse the notion of *coefficient* with the notion of *coordinate*: these are equivalent only when the divisor of the (closure) point is 1.

9.6.2 Member Enumeration Documentation

9.6.2.1 enum Parma_Polyhedra_Library::Generator::Type

The generator type.

Enumeration values:

LINE The generator is a line.

RAY The generator is a ray.

POINT The generator is a point.

CLOSURE_POINT The generator is a closure point.

9.6.3 Member Function Documentation

9.6.3.1 Generator line (const LinExpression & *e*) `[static]`

Shorthand for **Generator**(p. 83) **Generator::line(const LinExpression& e)**(p. 87).

Exceptions:

std::invalid_argument Thrown if the homogeneous part of `e` represents the origin of the vector space.

9.6.3.2 Generator ray (const LinExpression & *e*) `[static]`

Shorthand for **Generator**(p. 83) **Generator::ray(const LinExpression& e)**(p. 87).

Exceptions:

std::invalid_argument Thrown if the homogeneous part of `e` represents the origin of the vector space.

9.6.3.3 Generator point (const LinExpression & *e* = `LinExpression::zero()`, const Integer & *d* = `Integer_one()`) `[static]`

Shorthand for **Generator**(p. 83) **Generator::point(const LinExpression& e, const Integer& d)**(p. 87).

Both `e` and `d` are optional arguments, with default values **LinExpression::zero()**(p. 90) and Integer_one(), respectively.

Exceptions:

std::invalid_argument Thrown if `d` is zero.

9.6.3.4 Generator closure_point (const LinExpression & *e* = `LinExpression::zero()`, const Integer & *d* = `Integer_one()`) `[static]`

Shorthand for **Generator**(p. 83) **Generator::closure_point(const LinExpression& e, const Integer& d)**(p. 88).

Both `e` and `d` are optional arguments, with default values **LinExpression::zero()**(p. 90) and Integer_one(), respectively.

Exceptions:

std::invalid_argument Thrown if `d` is zero.

9.6.3.5 const Integer& Parma_Polyhedra_Library::Generator::coefficient (Variable *v*) const

Returns the coefficient of `v` in `*this`.

Exceptions:

std::invalid_argument Thrown if the index of `v` is greater than or equal to the space-dimension of `*this`.

9.6.3.6 const Integer& Parma_Polyhedra_Library::Generator::divisor () const

If `*this` is either a point or a closure point, returns its divisor.

Exceptions:

std::invalid_argument Thrown if `*this` is neither a point nor a closure point.

9.7 Parma_Polyhedra_Library::H79_Certificate Class Reference

A convergence certificate for the H79 widening operator.

Public Member Functions

- **H79_Certificate** ()
  Default constructor.
- **H79_Certificate** (const **Polyhedron** &ph)
  Constructor: computes the certificate for `ph`.
- **H79_Certificate** (const **H79_Certificate** &y)
  Copy constructor.
- ~**H79_Certificate** ()
  Destructor.
- int **compare** (const **H79_Certificate** &y) const
  The comparison function for certificates.
- int **compare** (const **Polyhedron** &ph) const
  Compares `*this` *with the certificate for polyhedron* `ph`.

9.7.1 Detailed Description

A convergence certificate for the H79 widening operator.

Convergence certificates are used to instantiate the BHZ03 framework so as to define widening operators for the finite powerset domain.

Note:
The convergence of the H79 widening can also be certified by **BHRZ03_Certificate**(p. 70).

9.7.2 Member Function Documentation

9.7.2.1 int Parma_Polyhedra_Library::H79_Certificate::compare (const H79_Certificate & *y*) const

The comparison function for certificates.

Returns:
-1, 0 or 1 depending on whether `*this` is smaller than, equal to, or greater than `y`, respectively.

Compares `*this` with `y`, using a total ordering which is a refinement of the limited growth ordering relation for the H79 widening.

9.8 Parma_Polyhedra_Library::H79_Certificate::Compare Struct Reference

A total ordering on H79 certificates.

Public Member Functions

- bool **operator()** (const **H79_Certificate** &x, const **H79_Certificate** &y) const

  Returns `true` *if and only if* `x` *comes before* `y`.

9.8.1 Detailed Description

A total ordering on H79 certificates.

This binary predicate defines a total ordering on H79 certificates which is used when storing information about sets of polyhedra.

9.9 Parma_Polyhedra_Library::LinExpression Class Reference

A linear expression.

Public Member Functions

- **LinExpression** ()

  Default constructor: returns a copy of **LinExpression::zero**()(p. 90).

- **LinExpression** (const **LinExpression** &e)

Ordinary copy-constructor.

- **∼LinExpression** ()

  Destructor.

- **LinExpression** (const **Integer** &n)

  Builds the linear expression corresponding to the inhomogeneous term `n`.

- **LinExpression** (const **Variable** v)

  Builds the linear expression corresponding to the variable `v`.

- **LinExpression** (const **Constraint** &c)

  Builds the linear expression corresponding to constraint `c`.

- **LinExpression** (const **Generator** &g)

  Builds the linear expression corresponding to generator `g` *(for points and closure points, the divisor is not copied).*

- dimension_type **space_dimension** () const

  Returns the dimension of the vector space enclosing `*this`.

- const **Integer** & **coefficient** (**Variable** v) const

  Returns the coefficient of `v` *in* `*this`.

- const **Integer** & **inhomogeneous_term** () const

  Returns the inhomogeneous term of `*this`.

Static Public Member Functions

- const **LinExpression** & **zero** ()

  Returns the (zero-dimension space) constant 0.

Related Functions

(Note that these are not member functions.)

- **LinExpression operator+** (const **LinExpression** &e1, const **LinExpression** &e2)

  Returns the linear expression `e1 + e2`.

- **LinExpression operator+** (const **Integer** &n, const **LinExpression** &e)

  Returns the linear expression `n + e`.

- **LinExpression operator+** (const **LinExpression** &e, const **Integer** &n)

  Returns the linear expression `e + n`.

- **LinExpression operator+** (const **LinExpression** &e)

  Returns the linear expression `e`.

- **LinExpression operator-** (const **LinExpression** &e)

  Returns the linear expression `- e`.

- **LinExpression operator-** (const **LinExpression** &e1, const **LinExpression** &e2)

  Returns the linear expression `e1 - e2`.

- **LinExpression operator-** (const **Integer** &n, const **LinExpression** &e)

  Returns the linear expression `n - e`.

- **LinExpression operator-** (const **LinExpression** &e, const **Integer** &n)

  Returns the linear expression `e - n`.

- **LinExpression operator** $*$ (const **Integer** &n, const **LinExpression** &e)

  Returns the linear expression `n * e`.

- **LinExpression operator** $*$ (const **LinExpression** &e, const **Integer** &n)

  Returns the linear expression `e * n`.

- **LinExpression** & **operator+=** (**LinExpression** &e1, const **LinExpression** &e2)

  Returns the linear expression `e1 + e2` *and assigns it to* `e1`.

- **LinExpression** & **operator+=** (**LinExpression** &e, const **Variable** v)

  Returns the linear expression `e + v` *and assigns it to* `e`.

- **LinExpression** & **operator+=** (**LinExpression** &e, const **Integer** &n)

  Returns the linear expression `e + n` *and assigns it to* `e`.

- **LinExpression** & **operator-=** (**LinExpression** &e1, const **LinExpression** &e2)

  Returns the linear expression `e1 - e2` *and assigns it to* `e1`.

- **LinExpression** & **operator-=** (**LinExpression** &e, const **Variable** v)

  Returns the linear expression `e - v` *and assigns it to* `e`.

- **LinExpression** & **operator-=** (**LinExpression** &e, const **Integer** &n)

  Returns the linear expression `e - n` *and assigns it to* `e`.

- **LinExpression** & **operator** $*$**=** (**LinExpression** &e, const **Integer** &n)

  Returns the linear expression `n * e` *and assigns it to* `e`.

- std::ostream & **operator**<< (std::ostream &s, const **LinExpression** &e)

  Output operator.

- void **swap** (**Parma_Polyhedra_Library::LinExpression** &x, **Parma_Polyhedra_Library::LinExpression** &y)

  Specializes `std::swap`.

9.9.1 Detailed Description

A linear expression.

An object of the class **LinExpression**(p. 89) represents the linear expression

$$\sum_{i=0}^{n-1} a_i x_i + b$$

where n is the dimension of the space, each a_i is the integer coefficient of the `i` -th variable x_i and b is the integer for the inhomogeneous term.

How to build a linear expression.

Linear expressions are the basic blocks for defining both constraints (i.e., linear equalities or inequalities) and generators (i.e., lines, rays, points and closure points). A full set of functions is defined to provide a convenient interface for building complex linear expressions starting from simpler ones and from objects of the classes **Variable**(p. 132) and Integer: available operators include unary negation, binary addition and subtraction, as well as multiplication by an Integer. The space-dimension of a linear expression is defined as the maximum space-dimension of the arguments used to build it: in particular, the space-dimension of a **Variable**(p. 132) `x` is defined as `x.id()+1`, whereas all the objects of the class Integer have space-dimension zero.

Example

The following code builds the linear expression $4x - 2y - z + 14$, having space-dimension 3:

```
LinExpression e = 4*x - 2*y - z + 14;
```

Another way to build the same linear expression is:

```
LinExpression e1 = 4*x;
LinExpression e2 = 2*y;
LinExpression e3 = z;
LinExpression e = LinExpression(14);
e += e1 - e2 - e3;
```

Note that `e1`, `e2` and `e3` have space-dimension 1, 2 and 3, respectively; also, in the fourth line of code, `e` is created with space-dimension zero and then extended to space-dimension 3.

9.9.2 Constructor & Destructor Documentation

9.9.2.1 Parma_Polyhedra_Library::LinExpression::LinExpression (const Constraint & *c*) [explicit]

Builds the linear expression corresponding to constraint `c`.

Given the constraint $c = \left(\sum_{i=0}^{n-1} a_i x_i + b \bowtie 0\right)$, where $\bowtie \in \{=, \geq, >\}$, builds the linear expression $\sum_{i=0}^{n-1} a_i x_i + b$. If `c` is an inequality (resp., equality) constraint, then the built linear expression is unique up to a positive (resp., non-zero) factor.

9.9.2.2 Parma_Polyhedra_Library::LinExpression::LinExpression (const Generator & *g*) [explicit]

Builds the linear expression corresponding to generator `g` (for points and closure points, the divisor is not copied).

Given the generator $g = (\frac{a_0}{d}, \ldots, \frac{a_{n-1}}{d})^{\mathrm{T}}$ (where, for lines and rays, we have $d = 1$), builds the linear expression $\sum_{i=0}^{n-1} a_i x_i$. The inhomogeneous term of the linear expression will always be 0. If `g` is a ray, point or closure point (resp., a line), then the linear expression is unique up to a positive (resp., non-zero) factor.

9.10 Parma_Polyhedra_Library::NNC_Polyhedron Class Reference

A not necessarily closed convex polyhedron.

Inherits **Parma_Polyhedra_Library::Polyhedron**.

Public Member Functions

- **NNC_Polyhedron** (dimension_type num_dimensions=0, **Degenerate_Kind** kind=UNIVERSE)

  Builds either the universe or the empty NNC polyhedron.

- **NNC_Polyhedron** (const ConSys &cs)

  Builds an NNC polyhedron from a system of constraints.

- **NNC_Polyhedron** (ConSys &cs)

  Builds an NNC polyhedron recycling a system of constraints.

- **NNC_Polyhedron** (const GenSys &gs)

  Builds an NNC polyhedron from a system of generators.

- **NNC_Polyhedron** (GenSys &gs)

  Builds an NNC polyhedron recycling a system of generators.

- **NNC_Polyhedron** (const **C_Polyhedron** &y)

  Builds an NNC polyhedron from the C polyhedron `y`.

- template<typename Box> **NNC_Polyhedron** (const Box &box, From_Bounding_Box dummy)

  Builds an NNC polyhedron out of a generic, interval-based bounding box.

- **NNC_Polyhedron** (const **NNC_Polyhedron** &y)

  Ordinary copy-constructor.

- **NNC_Polyhedron** & **operator=** (const **NNC_Polyhedron** &y)

  The assignment operator. (`*this` *and* `y` *can be dimension-incompatible.).*

- ~**NNC_Polyhedron** ()

  Destructor.

Static Public Member Functions

- dimension_type **max_space_dimension** ()

  Returns the maximum space dimension a **C_Polyhedron**(p. 71) *can handle.*

9.10.1 Detailed Description

A not necessarily closed convex polyhedron.

An object of the class **NNC_Polyhedron**(p. 93) represents a *not necessarily closed* (NNC) convex polyhedron in the vector space $\mathbb{R}^n$.

Note:

Since NNC polyhedra are a generalization of closed polyhedra, any object of the class **C_Polyhedron**(p. 71) can be (explicitly) converted into an object of the class **NNC_Polyhedron**(p. 93). The reason for defining two different classes is that objects of the class **C_Polyhedron**(p. 71) are characterized by a more efficient implementation, requiring less time and memory resources.

9.10.2 Constructor & Destructor Documentation

9.10.2.1 Parma_Polyhedra_Library::NNC_Polyhedron::NNC_Polyhedron (dimension_type *num_dimensions* = `0`, Degenerate_Kind *kind* = `UNIVERSE`) `[explicit]`

Builds either the universe or the empty NNC polyhedron.

Parameters:

num_dimensions The number of dimensions of the vector space enclosing the NNC polyhedron;

kind Specifies whether a universe or an empty NNC polyhedron should be built.

Exceptions:

std::length_error Thrown if `num_dimensions` exceeds the maximum allowed space dimension.

Both parameters are optional: by default, a 0-dimension space universe NNC polyhedron is built.

9.10.2.2 Parma_Polyhedra_Library::NNC_Polyhedron::NNC_Polyhedron (const ConSys & *cs*)

Builds an NNC polyhedron from a system of constraints.

The polyhedron inherits the space dimension of the constraint system.

Parameters:

cs The system of constraints defining the polyhedron. It is not declared `const` because its data-structures will be recycled to build the polyhedron.

9.10.2.3 Parma_Polyhedra_Library::NNC_Polyhedron::NNC_Polyhedron (ConSys & *cs*)

Builds an NNC polyhedron recycling a system of constraints.

The polyhedron inherits the space dimension of the constraint system.

Parameters:

cs The system of constraints defining the polyhedron. It is not declared `const` because its data-structures will be recycled to build the polyhedron.

9.10.2.4 Parma_Polyhedra_Library::NNC_Polyhedron::NNC_Polyhedron (const GenSys & *gs*)

Builds an NNC polyhedron from a system of generators.

The polyhedron inherits the space dimension of the generator system.

Parameters:

gs The system of generators defining the polyhedron. It is not declared `const` because its data-structures will be recycled to build the polyhedron.

Exceptions:

std::invalid_argument Thrown if the system of generators is not empty but has no points.

9.10.2.5 Parma_Polyhedra_Library::NNC_Polyhedron::NNC_Polyhedron (GenSys & *gs*)

Builds an NNC polyhedron recycling a system of generators.

The polyhedron inherits the space dimension of the generator system.

Parameters:

gs The system of generators defining the polyhedron. It is not declared `const` because its data-structures will be recycled to build the polyhedron.

Exceptions:

std::invalid_argument Thrown if the system of generators is not empty but has no points.

9.10.2.6 template<typename Box> Parma_Polyhedra_Library::NNC_Polyhedron::NNC_-Polyhedron (const Box & *box*, From_Bounding_Box *dummy*)

Builds an NNC polyhedron out of a generic, interval-based bounding box.

For a description of the methods that should be provided by the template class Box, see the documentation of the protected method: template <typename Box> **Polyhedron::Polyhedron(Topology topol, const Box& box)**(p. 114);

Parameters:

box The bounding box representing the polyhedron to be built;

dummy A dummy tag to syntactically differentiate this one from the other constructors.

9.11 Parma_Polyhedra_Library::Poly_Con_Relation Class Reference

The relation between a polyhedron and a constraint.

Public Member Functions

- bool **implies** (const **Poly_Con_Relation** &y) const

  True if and only if `*this` *implies* `y`.

- bool **OK** () const

  Checks if all the invariants are satisfied.

Static Public Member Functions

- **Poly_Con_Relation nothing** ()
  The assertion that says nothing.
- **Poly_Con_Relation is_disjoint** ()
  The polyhedron and the set of points satisfying the constraint are disjoint.
- **Poly_Con_Relation strictly_intersects** ()
  The polyhedron intersects the set of points satisfying the constraint, but it is not included in it.
- **Poly_Con_Relation is_included** ()
  The polyhedron is included in the set of points satisfying the constraint.
- **Poly_Con_Relation saturates** ()
  The polyhedron is included in the set of points saturating the constraint.

Related Functions

(Note that these are not member functions.)

- bool **operator==** (const **Poly_Con_Relation** &x, const **Poly_Con_Relation** &y)
  True if and only if `x` *and* `y` *are logically equivalent.*
- bool **operator!=** (const **Poly_Con_Relation** &x, const **Poly_Con_Relation** &y)
  True if and only if `x` *and* `y` *are not logically equivalent.*
- **Poly_Con_Relation operator &&** (const **Poly_Con_Relation** &x, const **Poly_Con_Relation** &y)
  Yields the logical conjunction of `x` *and* `y`*.*
- **Poly_Con_Relation operator-** (const **Poly_Con_Relation** &x, const **Poly_Con_Relation** &y)
  Yields the assertion with all the conjuncts of `x` *that are not in* `y`*.*
- std::ostream & **operator**<< (std::ostream &s, const **Poly_Con_Relation** &r)
  Output operator.

9.11.1 Detailed Description

The relation between a polyhedron and a constraint.

This class implements conjunctions of assertions on the relation between a polyhedron and a constraint.

9.12 Parma_Polyhedra_Library::Poly_Gen_Relation Class Reference

The relation between a polyhedron and a generator.

Public Member Functions

- bool **implies** (const **Poly_Gen_Relation** &y) const
  True if and only if `*this` *implies* `y`.
- bool **OK** () const
  Checks if all the invariants are satisfied.

Static Public Member Functions

- **Poly_Gen_Relation nothing** ()
  The assertion that says nothing.
- **Poly_Gen_Relation subsumes** ()
  Adding the generator would not change the polyhedron.

Related Functions

(Note that these are not member functions.)

- bool **operator==** (const **Poly_Gen_Relation** &x, const **Poly_Gen_Relation** &y)
  True if and only if `x` *and* `y` *are logically equivalent.*
- bool **operator!=** (const **Poly_Gen_Relation** &x, const **Poly_Gen_Relation** &y)
  True if and only if `x` *and* `y` *are not logically equivalent.*
- **Poly_Gen_Relation operator &&** (const **Poly_Gen_Relation** &x, const **Poly_Gen_Relation** &y)
  Yields the logical conjunction of `x` *and* `y`.
- **Poly_Gen_Relation operator-** (const **Poly_Gen_Relation** &x, const **Poly_Gen_Relation** &y)
  Yields the assertion with all the conjuncts of `x` *that are not in* `y`.
- std::ostream & **operator**<< (std::ostream &s, const **Poly_Gen_Relation** &r)
  Output operator.

9.12.1 Detailed Description

The relation between a polyhedron and a generator.

This class implements conjunctions of assertions on the relation between a polyhedron and a generator.

9.13 Parma_Polyhedra_Library::Polyhedra_PowerSet< PH > Class Template Reference

The powerset construction instantiated on PPL polyhedra.

Inherits **Parma_Polyhedra_Library::PowerSet< Parma_Polyhedra_Library::Determinate< PH > >**.

Public Member Functions

Constructors

- **Polyhedra_PowerSet** (dimension_type num_dimensions=0, **Polyhedron::Degenerate_Kind** kind=Polyhedron::UNIVERSE)

  Builds a universe (top) or empty (bottom) **Polyhedra_PowerSet**(p. 97).

- **Polyhedra_PowerSet** (const **Polyhedra_PowerSet** &y)

  Ordinary copy-constructor.

- **Polyhedra_PowerSet** (const ConSys &cs)

  Creates a **Polyhedra_PowerSet**(p. 97) *with the same information contents as* `cs`.

Member Functions that Do Not Modify the Powerset of Polyhedra

- dimension_type **space_dimension** () const

  Returns the dimension of the vector space enclosing `*this`.

- bool **geometrically_covers** (const **Polyhedra_PowerSet** &y) const

  Returns `true` *if and only if* `*this` *geometrically covers* `y`*, i.e., if any point (in some element) of* `y` *is also a point (of some element) of* `*this`.

- bool **geometrically_equals** (const **Polyhedra_PowerSet** &y) const

  Returns `true` *if and only if* `*this` *is geometrically equal to* `y`*, i.e., if (the elements of)* `*this` *and* `y` *contain the same set of points.*

- bool **OK** () const

  Checks if all the invariants are satisfied.

Space-Dimension Preserving Member Functions that May Modify the Powerset of Polyhedra

- void **add_constraint** (const **Constraint** &c)

  Intersects `*this` *with constraint* `c`.

- bool **add_constraint_and_minimize** (const **Constraint** &c)

  Intersects `*this` *with the constraint* `c`*, minimizing the result.*

- void **add_constraints** (const ConSys &cs)

  Intersects `*this` *with the constraints in* `cs`.

- bool **add_constraints_and_minimize** (const ConSys &cs)

  Intersects `*this` *with the constraints in* `cs`*, minimizing the result.*

- void **pairwise_reduce** ()

  Assign to `*this` *the result of (recursively) merging together the pairs of polyhedra whose poly-hull is the same as their set-theoretical union.*

- template<typename Widening> void **BGP99_extrapolation_assign** (const **Polyhedra_PowerSet** &y, Widening wf, unsigned max_disjuncts)

  Assigns to `*this` *the result of applying the BGP99 extrapolation operator to* `*this` *and* `y`*, using the widening function* `wf` *and the cardinality threshold* `max_disjuncts`.

- template<typename Cert, typename Widening> void **BHZ03_widening_assign** (const **Polyhedra_PowerSet** &y, Widening wf)

  Assigns to `*this` *the result of computing the* **BHZ03-widening**(p. 13) *between* `*this` *and* `y`*, using the widening function* `wf` *certified by the convergence certificate* `Cert`.

- template<typename Widening> void **BHZ03_widening_assign** (const **Polyhedra_PowerSet** &y, Widening wf)

  An instance of the BHZ03 framework using the widening function `wf` *certified by* **BHRZ03_-Certificate**(p. 70).

Member Functions that May Modify the Dimension of the Vector Space

- **Polyhedra_PowerSet** & **operator=** (const **Polyhedra_PowerSet** &y)

  The assignment operator (`*this` *and* `y` *can be dimension-incompatible).*

- void **swap** (**Polyhedra_PowerSet** &y)

  Swaps `*this` *with* `y`.

- void **add_dimensions_and_embed** (dimension_type m)

  Adds `m` *new dimensions to the space containing* `*this` *and embeds each polyhedron in* `*this` *in the new space.*

- void **add_dimensions_and_project** (dimension_type m)

  For each polyhedron in `*this`*, adds* `m` *new dimensions without embedding it in the new space.*

- void **concatenate_assign** (const **Polyhedra_PowerSet** &y)

  Assigns to `*this` *the concatenation of* `*this` *and* `y`.

- void **remove_dimensions** (const **Variables_Set** &to_be_removed)

  Removes all the specified dimensions.

- void **remove_higher_dimensions** (dimension_type new_dimension)

  Removes the higher dimensions so that the resulting space will have dimension `new_dimension`.

- template<typename PartialFunction> void **map_dimensions** (const PartialFunction &pfunc)

  Remaps the dimensions of the vector space according to a partial function.

Static Public Member Functions

- dimension_type **max_space_dimension** ()

  Returns the maximum space dimension a Polyhedra_Powerset<PH> can handle.

Related Functions

(Note that these are not member functions.)

- Widening_Function< PH > **widen_fun** (void(PH::*wm)(const PH &, unsigned *))

  Wraps a widening method into a function object.

- Limited_Widening_Function< PH > **widen_fun** (void(PH::∗lwm)(const PH &, const ConSys &, unsigned ∗), const ConSys &cs)

    Wraps a limited widening method into a function object.

- void **swap** (**Parma_Polyhedra_Library::Polyhedra_PowerSet**< PH > &x, **Parma_Polyhedra_-Library::Polyhedra_PowerSet**< PH > &y)

    Specializes `std::swap`.

- std::pair< PH, **Polyhedra_PowerSet**< **NNC_Polyhedron** > > **linear_partition** (const PH &p, const PH &q)

    Partitions `q` *with respect to* `p`.

9.13.1 Detailed Description

template<typename PH> class Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >

The powerset construction instantiated on PPL polyhedra.

9.13.2 Constructor & Destructor Documentation

9.13.2.1 template<typename PH> Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::Polyhedra_PowerSet (dimension_type *num_dimensions* = `0`, Polyhedron::Degenerate_Kind *kind* = `Polyhedron::UNIVERSE`) `[explicit]`

Builds a universe (top) or empty (bottom) **Polyhedra_PowerSet**(p. 97).

Parameters:

num_dimensions The number of dimensions of the vector space enclosing the powerset;

kind Specifies whether the universe or the empty powerset has to be built.

9.13.3 Member Function Documentation

9.13.3.1 template<typename PH> bool Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::geometrically_covers (const Polyhedra_PowerSet< PH > & *y*) const

Returns `true` if and only if `*this` geometrically covers `y`, i.e., if any point (in some element) of `y` is also a point (of some element) of `*this`.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

Warning:

This may be *really* expensive!

9.13.3.2 template<typename PH> bool Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::geometrically_equals (const Polyhedra_PowerSet< PH > & *y*) const

Returns `true` if and only if `*this` is geometrically equal to `y`, i.e., if (the elements of) `*this` and `y` contain the same set of points.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

Warning:

This may be *really* expensive!

9.13.3.3 template<typename PH> void Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::add_constraint (const Constraint & *c*)

Intersects `*this` with constraint `c`.

Exceptions:

std::invalid_argument Thrown if `*this` and constraint `c` are topology-incompatible or dimension-incompatible.

9.13.3.4 template<typename PH> bool Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::add_constraint_and_minimize (const Constraint & *c*)

Intersects `*this` with the constraint `c`, minimizing the result.

Returns:

`false` if and only if the result is empty.

Exceptions:

std::invalid_argument Thrown if `*this` and `c` are topology-incompatible or dimension-incompatible.

9.13.3.5 template<typename PH> void Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::add_constraints (const ConSys & *cs*)

Intersects `*this` with the constraints in `cs`.

Parameters:

cs The constraints to intersect with.

Exceptions:

std::invalid_argument Thrown if `*this` and `cs` are topology-incompatible or dimension-incompatible.

9.13.3.6 template<typename PH> bool Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::add_constraints_and_minimize (const ConSys & *cs*)

Intersects `*this` with the constraints in `cs`, minimizing the result.

Returns:

`false` if and only if the result is empty.

Parameters:

cs The constraints to intersect with.

Exceptions:

std::invalid_argument Thrown if `*this` and `cs` are topology-incompatible or dimension-incompatible.

9.13.3.7 template<typename PH> void Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::pairwise_reduce ()

Assign to `*this` the result of (recursively) merging together the pairs of polyhedra whose poly-hull is the same as their set-theoretical union.

On exit, for all the pairs $\mathcal{P}$, $\mathcal{Q}$ of different polyhedra in `*this`, we have $\mathcal{P} \uplus \mathcal{Q} \neq \mathcal{P} \cup \mathcal{Q}$.

9.13.3.8 template<typename PH> template<typename Widening> void Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::BGP99_extrapolation_assign (const Polyhedra_PowerSet< PH > & *y*, Widening *wf*, unsigned *max_disjuncts*)

Assigns to `*this` the result of applying the BGP99 extrapolation operator to `*this` and `y`, using the widening function `wf` and the cardinality threshold `max_disjuncts`.

Parameters:

y A finite powerset of polyhedra. It *must* definitely entail `*this`;

wf The widening function to be used on polyhedra objects. It is obtained from the corresponding widening method by using the helper function Parma_Polyhedra_Library::widen_fun. Legal values are, e.g., `widen_fun(&Polyhedron::H79_widening_assign)` and `widen_fun(&Polyhedron::limited_H79_extrapolation_assign, cs)`;

max_disjuncts The maximum number of disjuncts occurring in the powerset `*this` *before* starting the computation. If this number is exceeded, some of the disjuncts in `*this` are collapsed (i.e., joined together).

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

For a description of the extrapolation operator, see **[BGP99]**(p. 15) and **[BHZ03b]**(p. 15).

9.13.3.9 template<typename PH> template<typename Cert, typename Widening> void Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::BHZ03_widening_assign (const Polyhedra_PowerSet< PH > & *y*, Widening *wf*)

Assigns to `*this` the result of computing the **BHZ03-widening**(p. 13) between `*this` and `y`, using the widening function `wf` certified by the convergence certificate `Cert`.

Parameters:

y The finite powerset of polyhedra computed in the previous iteration step. It *must* definitely entail `*this`;

wf The widening function to be used on polyhedra objects. It is obtained from the corresponding widening method by using the helper function widen_fun. Legal values are, e.g., `widen_fun(&Polyhedron::H79_widening_assign)` and `widen_fun(&Polyhedron::limited_H79_extrapolation_assign, cs)`.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

Warning:

In order to obtain a proper widening operator, the template parameter `Cert` should be a finite convergence certificate for the base-level widening function `wf`; otherwise, an extrapolation operator is obtained. For a description of the methods that should be provided by `Cert`, see **BHRZ03_Certificate**(p. 70) or **H79_Certificate**(p. 88).

9.13.3.10 template<typename PH> void Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::concatenate_assign (const Polyhedra_PowerSet< PH > & *y*)

Assigns to `*this` the concatenation of `*this` and `y`.

The result is obtained by computing the pairwise **concatenation**(p. 6) of each polyhedron in `*this` with each polyhedron in `y`.

9.13.3.11 template<typename PH> void Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::remove_dimensions (const Variables_Set & *to_be_removed*)

Removes all the specified dimensions.

Parameters:

to_be_removed The set of **Variable**(p. 132) objects corresponding to the dimensions to be removed.

Exceptions:

std::invalid_argument Thrown if `*this` is dimension-incompatible with one of the **Variable**(p. 132) objects contained in `to_be_removed`.

9.13.3.12 template<typename PH> void Parma_Polyhedra_Library::Polyhedra_PowerSet< PH >::remove_higher_dimensions (dimension_type *new_dimension*)

Removes the higher dimensions so that the resulting space will have dimension `new_dimension`.

Exceptions:

std::invalid_argument Thrown if `new_dimensions` is greater than the space dimension of `*this`.

9.13.3.13 template<typename PH> template<typename PartialFunction> void Parma_-Polyhedra_Library::Polyhedra_PowerSet< PH >::map_dimensions (const PartialFunction & *pfunc*)

Remaps the dimensions of the vector space according to a partial function.

See also **Polyhedron::map_dimensions**(p. 128).

9.13.4 Friends And Related Function Documentation

9.13.4.1 template<typename PH> Widening_Function< PH > widen_fun (void(PH::* *wm*)(const PH &, unsigned *)) `[related]`

Wraps a widening method into a function object.

Parameters:

wm The widening method.

9.13.4.2 template<typename PH> Limited_Widening_Function< PH > widen_fun (void(PH::* *lwm*)(const PH &, const ConSys &, unsigned *), const ConSys & *cs*) `[related]`

Wraps a limited widening method into a function object.

Parameters:

lwm The limited widening method.

cs The constraint system limiting the widening.

9.13.4.3 template<typename PH> std::pair< PH, Polyhedra_PowerSet< NNC_Polyhedron > > linear_partition (const PH & *p*, const PH & *q*) [related]

Partitions `q` with respect to `p`.

Let `p` and `q` be two polyhedra. The function returns an object `r` of type `std::pair<PH,` **Polyhedra_PowerSet** (p. 97) <**NNC_Polyhedron** (p. 93) > > such that

- `r.first` is the intersection of `p` and `q`;
- `r.second` has the property that all its elements are not empty, pairwise disjoint, and disjoint from `p`;
- the union of `r.first` with all the elements of `r.second` gives `q` (i.e., `r` is the representation of a partition of `q`).

9.14 Parma_Polyhedra_Library::Polyhedron Class Reference

The base class for convex polyhedra.

Inherited by **Parma_Polyhedra_Library::C_Polyhedron**, and **Parma_Polyhedra_Library::NNC_Polyhedron**.

Public Types

- enum **Degenerate_Kind** { **UNIVERSE**, **EMPTY** }

  Kinds of degenerate polyhedra.

Public Member Functions

Member Functions that Do Not Modify the Polyhedron

- dimension_type **space_dimension** () const

  Returns the dimension of the vector space enclosing `*this`.
- const ConSys & **constraints** () const

  Returns the system of constraints.
- const ConSys & **minimized_constraints** () const

  Returns the system of constraints, with no redundant constraint.
- const GenSys & **generators** () const

  Returns the system of generators.
- const GenSys & **minimized_generators** () const

  Returns the system of generators, with no redundant generator.
- **Poly_Con_Relation relation_with** (const **Constraint** &c) const

  Returns the relations holding between the polyhedron `*this` *and the constraint* `c`.
- **Poly_Gen_Relation relation_with** (const **Generator** &g) const

  Returns the relations holding between the polyhedron `*this` *and the generator* `g`.

- bool **is_empty** () const
  Returns `true` *if and only if* `*this` *is an empty polyhedron.*

- bool **is_universe** () const
  Returns `true` *if and only if* `*this` *is a universe polyhedron.*

- bool **is_topologically_closed** () const
  Returns `true` *if and only if* `*this` *is a topologically closed subset of the vector space.*

- bool **is_disjoint_from** (const **Polyhedron** &y) const
  Returns `true` *if and only if* `*this` *and* `y` *are disjoint.*

- bool **is_bounded** () const
  Returns `true` *if and only if* `*this` *is a bounded polyhedron.*

- bool **bounds_from_above** (const **LinExpression** &expr) const
  Returns `true` *if and only if* `expr` *is bounded from above in* `*this`.

- bool **bounds_from_below** (const **LinExpression** &expr) const
  Returns `true` *if and only if* `expr` *is bounded from below in* `*this`.

- bool **maximize** (const **LinExpression** &expr, **Integer** &sup_n, **Integer** &sup_d, bool &maximum) const
  Returns `true` *if and only if* `*this` *is not empty and* `expr` *is bounded from above in* `*this`*, in which case the supremum value is computed.*

- bool **maximize** (const **LinExpression** &expr, **Integer** &sup_n, **Integer** &sup_d, bool &maximum, const **Generator** **const pppoint) const
  Returns `true` *if and only if* `*this` *is not empty and* `expr` *is bounded from above in* `*this`*, in which case the supremum value and a point where* `expr` *reaches it are computed.*

- bool **minimize** (const **LinExpression** &expr, **Integer** &inf_n, **Integer** &inf_d, bool &minimum) const
  Returns `true` *if and only if* `*this` *is not empty and* `expr` *is bounded from below in* `*this`*, in which case the infimum value is computed.*

- bool **minimize** (const **LinExpression** &expr, **Integer** &inf_n, **Integer** &inf_d, bool &minimum, const **Generator** **const pppoint) const
  Returns `true` *if and only if* `*this` *is not empty and* `expr` *is bounded from below in* `*this`*, in which case the infimum value and a point where* `expr` *reaches it are computed.*

- bool **contains** (const **Polyhedron** &y) const
  Returns `true` *if and only if* `*this` *contains* `y`.

- bool **strictly_contains** (const **Polyhedron** &y) const
  Returns `true` *if and only if* `*this` *strictly contains* `y`.

- template<typename Box> void **shrink_bounding_box** (Box &box, Complexity_Class complexity=ANY) const
  Uses `*this` *to shrink a generic, interval-based bounding box.*

- bool **OK** (bool check_not_empty=false) const
  Checks if all the invariants are satisfied.

Space-Dimension Preserving Member Functions that May Modify the Polyhedron

- void **add_constraint** (const **Constraint** &c)
  Adds a copy of constraint `c` *to the system of constraints of* `*this` *(without minimizing the result).*
- bool **add_constraint_and_minimize** (const **Constraint** &c)
  Adds a copy of constraint `c` *to the system of constraints of* `*this`*, minimizing the result.*
- void **add_generator** (const **Generator** &g)
  Adds a copy of generator `g` *to the system of generators of* `*this` *(without minimizing the result).*
- bool **add_generator_and_minimize** (const **Generator** &g)
  Adds a copy of generator `g` *to the system of generators of* `*this`*, minimizing the result.*
- void **add_constraints** (const ConSys &cs)
  Adds a copy of the constraints in `cs` *to the system of constraints of* `*this` *(without minimizing the result).*
- void **add_recycled_constraints** (ConSys &cs)
  Adds the constraints in `cs` *to the system of constraints of* `*this` *(without minimizing the result).*
- bool **add_constraints_and_minimize** (const ConSys &cs)
  Adds a copy of the constraints in `cs` *to the system of constraints of* `*this`*, minimizing the result.*
- bool **add_recycled_constraints_and_minimize** (ConSys &cs)
  Adds the constraints in `cs` *to the system of constraints of* `*this`*, minimizing the result.*
- void **add_generators** (const GenSys &gs)
  Adds a copy of the generators in `gs` *to the system of generators of* `*this` *(without minimizing the result).*
- void **add_recycled_generators** (GenSys &gs)
  Adds the generators in `gs` *to the system of generators of* `*this` *(without minimizing the result).*
- bool **add_generators_and_minimize** (const GenSys &gs)
  Adds a copy of the generators in `gs` *to the system of generators of* `*this`*, minimizing the result.*
- bool **add_recycled_generators_and_minimize** (GenSys &gs)
  Adds the generators in `gs` *to the system of generators of* `*this`*, minimizing the result.*
- void **intersection_assign** (const **Polyhedron** &y)
  Assigns to `*this` *the intersection of* `*this` *and* `y`*. The result is not guaranteed to be minimized.*
- bool **intersection_assign_and_minimize** (const **Polyhedron** &y)
  Assigns to `*this` *the intersection of* `*this` *and* `y`*, minimizing the result.*
- void **poly_hull_assign** (const **Polyhedron** &y)
  Assigns to `*this` *the poly-hull of* `*this` *and* `y`*. The result is not guaranteed to be minimized.*
- bool **poly_hull_assign_and_minimize** (const **Polyhedron** &y)
  Assigns to `*this` *the poly-hull of* `*this` *and* `y`*, minimizing the result.*
- void **poly_difference_assign** (const **Polyhedron** &y)
  Assigns to `*this` *the* **poly-difference**(p. 6) *of* `*this` *and* `y`*. The result is not guaranteed to be minimized.*

- void **affine_image** (**Variable** var, const **LinExpression** &expr, const **Integer** &denominator=Integer_one())

  Assigns to `*this` *the* **affine image**(p. 8) *of* `*this` *under the function mapping variable* `var` *to the affine expression specified by* `expr` *and* `denominator`.

- void **affine_preimage** (**Variable** var, const **LinExpression** &expr, const **Integer** &denominator=Integer_one())

  Assigns to `*this` *the* **affine preimage**(p. 8) *of* `*this` *under the function mapping variable* `var` *to the affine expression specified by* `expr` *and* `denominator`.

- void **generalized_affine_image** (**Variable** var, const Relation_Symbol relsym, const **LinExpression** &expr, const **Integer** &denominator=Integer_one())

  Assigns to `*this` *the image of* `*this` *with respect to the* **generalized affine transfer function**(p. 9) $\mathrm{var}' \bowtie \frac{\mathrm{expr}}{\mathrm{denominator}}$, *where* $\bowtie$ *is the relation symbol encoded by* `relsym`.

- void **generalized_affine_image** (const **LinExpression** &lhs, const Relation_Symbol relsym, const **LinExpression** &rhs)

  Assigns to `*this` *the image of* `*this` *with respect to the* **generalized affine transfer function**(p. 9) $\mathrm{lhs}' \bowtie \mathrm{rhs}$, *where* $\bowtie$ *is the relation symbol encoded by* `relsym`.

- void **time_elapse_assign** (const **Polyhedron** &y)

  Assigns to `*this` *the result of computing the* **time-elapse**(p. 9) *between* `*this` *and* `y`.

- void **topological_closure_assign** ()

  Assigns to `*this` *its topological closure.*

- void **BHRZ03_widening_assign** (const **Polyhedron** &y, unsigned *tp=0)

  Assigns to `*this` *the result of computing the* **BHRZ03-widening**(p. 10) *between* `*this` *and* `y`.

- void **limited_BHRZ03_extrapolation_assign** (const **Polyhedron** &y, const ConSys &cs, unsigned *tp=0)

  Improves the result of the **BHRZ03-widening**(p. 10) *computation by also enforcing those constraints in* `cs` *that are satisfied by all the points of* `*this`.

- void **bounded_BHRZ03_extrapolation_assign** (const **Polyhedron** &y, const ConSys &cs, unsigned *tp=0)

  Improves the result of the **BHRZ03-widening**(p. 10) *computation by also enforcing those constraints in* `cs` *that are satisfied by all the points of* `*this`, *plus all the constraints of the form* $\pm x \leq r$ *and* $\pm x < r$, *with* $r \in \mathbb{Q}$, *that are satisfied by all the points of* `*this`.

- void **H79_widening_assign** (const **Polyhedron** &y, unsigned *tp=0)

  Assigns to `*this` *the result of computing the* **H79-widening**(p. 10) *between* `*this` *and* `y`.

- void **limited_H79_extrapolation_assign** (const **Polyhedron** &y, const ConSys &cs, unsigned *tp=0)

  Improves the result of the **H79-widening**(p. 10) *computation by also enforcing those constraints in* `cs` *that are satisfied by all the points of* `*this`.

- void **bounded_H79_extrapolation_assign** (const **Polyhedron** &y, const ConSys &cs, unsigned *tp=0)

  Improves the result of the **H79-widening**(p. 10) *computation by also enforcing those constraints in* `cs` *that are satisfied by all the points of* `*this`, *plus all the constraints of the form* $\pm x \leq r$ *and* $\pm x < r$, *with* $r \in \mathbb{Q}$, *that are satisfied by all the points of* `*this`.

Member Functions that May Modify the Dimension of the Vector Space

- void **add_dimensions_and_embed** (dimension_type m)

  Adds `m` *new dimensions and embeds the old polyhedron in the new space.*

- void **add_dimensions_and_project** (dimension_type m)

  Adds `m` *new dimensions to the polyhedron and does not embed it in the new space.*

- void **concatenate_assign** (const **Polyhedron** &y)

  Assigns to `*this` *the* **concatenation**(p. 6) *of* `*this` *and* `y`*, taken in this order.*

- void **remove_dimensions** (const **Variables_Set** &to_be_removed)

  Removes all the specified dimensions.

- void **remove_higher_dimensions** (dimension_type new_dimension)

  Removes the higher dimensions so that the resulting space will have dimension `new_dimension`.

- template<typename PartialFunction> void **map_dimensions** (const PartialFunction &pfunc)

  Remaps the dimensions of the vector space according to a **partial function**(p. 7).

- void **expand_dimension** (**Variable** var, dimension_type m)

  Creates `m` *copies of the dimension corresponding to* `var`.

- void **fold_dimensions** (const **Variables_Set** &to_be_folded, **Variable** var)

  Folds the dimensions in `to_be_folded` *into* `var`.

Miscellaneous Member Functions

- ~**Polyhedron** ()

  Destructor.

- void **swap** (**Polyhedron** &y)

  Swaps `*this` *with polyhedron* `y`*. (*`*this` *and* `y` *can be dimension-incompatible.).*

Static Public Member Functions

- dimension_type **max_space_dimension** ()

  Returns the maximum space dimension all kinds of **Polyhedron**(p. 104) *can handle.*

Protected Member Functions

- **Polyhedron** (Topology topol, dimension_type num_dimensions, **Degenerate_Kind** kind)

  Builds a polyhedron having the specified properties.

- **Polyhedron** (const **Polyhedron** &y)

  Ordinary copy-constructor.

- **Polyhedron** (Topology topol, const ConSys &cs)

  Builds a polyhedron from a system of constraints.

- **Polyhedron** (Topology topol, ConSys &cs)

Builds a polyhedron recycling a system of constraints.

- **Polyhedron** (Topology topol, const GenSys &gs)

  Builds a polyhedron from a system of generators.

- **Polyhedron** (Topology topol, GenSys &gs)

  Builds a polyhedron recycling a system of generators.

- template<typename Box> **Polyhedron** (Topology topol, const Box &box)

  Builds a polyhedron out of a generic, interval-based bounding box.

- **Polyhedron** & **operator=** (const **Polyhedron** &y)

  The assignment operator. (`*this` and `y` can be dimension-incompatible.).

Related Functions

(Note that these are not member functions.)

- std::ostream & **operator**<< (std::ostream &s, const **Polyhedron** &ph)

  Output operator.

- bool **operator==** (const **Polyhedron** &x, const **Polyhedron** &y)

  Returns `true` if and only if `x` and `y` are the same polyhedron.

- bool **operator!=** (const **Polyhedron** &x, const **Polyhedron** &y)

  Returns `true` if and only if `x` and `y` are different polyhedra.

- void **swap** (**Parma_Polyhedra_Library::Polyhedron** &x, **Parma_Polyhedra_Library::Polyhedron** &y)

  Specializes `std::swap`.

- template<typename PH> bool **poly_hull_assign_if_exact** (PH &p, const PH &q)

  If the poly-hull between `p` and `q` is exact it is assigned to `p`.

9.14.1 Detailed Description

The base class for convex polyhedra.

An object of the class **Polyhedron**(p. 104) represents a convex polyhedron in the vector space $\mathbb{R}^n$.

A polyhedron can be specified as either a finite system of constraints or a finite system of generators (see Section **Representations of Convex Polyhedra**(p. 3)) and it is always possible to obtain either representation. That is, if we know the system of constraints, we can obtain from this the system of generators that define the same polyhedron and vice versa. These systems can contain redundant members: in this case we say that they are not in the minimal form. Most operators on polyhedra are provided with two implementations: one of these, denoted `<operator-name>_and_minimize`, also enforces the minimization of the representations, and returns the Boolean value `false` whenever the resulting polyhedron turns out to be empty.

Two key attributes of any polyhedron are its topological kind (recording whether it is a **C_-Polyhedron**(p. 71) or an **NNC_Polyhedron**(p. 93) object) and its space dimension (the dimension $n \in \mathbb{N}$ of the enclosing vector space):

- all polyhedra, the empty ones included, are endowed with a specific topology and space dimension;
- most operations working on a polyhedron and another object (i.e., another polyhedron, a constraint or generator, a set of variables, etc.) will throw an exception if the polyhedron and the object are not both topology-compatible and dimension-compatible (see Section **Representations of Convex Polyhedra**(p. 3));
- there is no way to change the topology of a polyhedron; rather, there are constructors of the two derived classes that builds a new polyhedron having a topology when provided with the corresponding polyhedron of the other topology;
- the only ways to change the space dimension of a polyhedron are:
  - *explicit* calls to operators provided for that purpose;
  - standard copy, assignment and swap operators.

Note that four different polyhedra can be defined on the zero-dimension space: the empty polyhedron, either closed or NNC, and the universe polyhedron R^0, again either closed or NNC.

In all the examples it is assumed that variables `x` and `y` are defined (where they are used) as follows:

```
Variable x(0);
Variable y(1);
```

Example 1

The following code builds a polyhedron corresponding to a square in $\mathbb{R}^2$, given as a system of constraints:

```
ConSys cs;
cs.insert(x >= 0);
cs.insert(x <= 3);
cs.insert(y >= 0);
cs.insert(y <= 3);
C_Polyhedron ph(cs);
```

The following code builds the same polyhedron as above, but starting from a system of generators specifying the four vertices of the square:

```
GenSys gs;
gs.insert(point(0*x + 0*y));
gs.insert(point(0*x + 3*y));
gs.insert(point(3*x + 0*y));
gs.insert(point(3*x + 3*y));
C_Polyhedron ph(gs);
```

Example 2

The following code builds an unbounded polyhedron corresponding to a half-strip in $\mathbb{R}^2$, given as a system of constraints:

```
ConSys cs;
cs.insert(x >= 0);
cs.insert(x - y <= 0);
cs.insert(x - y + 1 >= 0);
C_Polyhedron ph(cs);
```

The following code builds the same polyhedron as above, but starting from the system of generators specifying the two vertices of the polyhedron and one ray:

```
GenSys gs;
gs.insert(point(0*x + 0*y));
gs.insert(point(0*x + y));
gs.insert(ray(x - y));
C_Polyhedron ph(gs);
```

Example 3

The following code builds the polyhedron corresponding to an half-plane by adding a single constraint to the universe polyhedron in $\mathbb{R}^2$:

```
C_Polyhedron ph(2);
ph.add_constraint(y >= 0);
```

The following code builds the same polyhedron as above, but starting from the empty polyhedron in the space $\mathbb{R}^2$ and inserting the appropriate generators (a point, a ray and a line).

```
C_Polyhedron ph(2, Polyhedron::EMPTY);
ph.add_generator(point(0*x + 0*y));
ph.add_generator(ray(y));
ph.add_generator(line(x));
```

Note that, although the above polyhedron has no vertices, we must add one point, because otherwise the result of the Minkowsky's sum would be an empty polyhedron. To avoid subtle errors related to the minimization process, it is required that the first generator inserted in an empty polyhedron is a point (otherwise, an exception is thrown).

Example 4

The following code shows the use of the function `add_dimensions_and_embed`:

```
C_Polyhedron ph(1);
ph.add_constraint(x == 2);
ph.add_dimensions_and_embed(1);
```

We build the universe polyhedron in the 1-dimension space $\mathbb{R}$. Then we add a single equality constraint, thus obtaining the polyhedron corresponding to the singleton set $\{2\} \subseteq \mathbb{R}$. After the last line of code, the resulting polyhedron is

$$\left\{\, (2, y)^{\mathrm{T}} \in \mathbb{R}^2 \;\middle|\; y \in \mathbb{R} \,\right\}.$$

Example 5

The following code shows the use of the function `add_dimensions_and_project`:

```
C_Polyhedron ph(1);
ph.add_constraint(x == 2);
ph.add_dimensions_and_project(1);
```

The first two lines of code are the same as in Example 4 for `add_dimensions_and_embed`. After the last line of code, the resulting polyhedron is the singleton set $\left\{(2, 0)^{\mathrm{T}}\right\} \subseteq \mathbb{R}^2$.

Example 6

The following code shows the use of the function `affine_image`:

```
C_Polyhedron ph(2, Polyhedron::EMPTY);
ph.add_generator(point(0*x + 0*y));
ph.add_generator(point(0*x + 3*y));
ph.add_generator(point(3*x + 0*y));
ph.add_generator(point(3*x + 3*y));
LinExpression coeff = x + 4;
ph.affine_image(x, coeff);
```

In this example the starting polyhedron is a square in $\mathbb{R}^2$, the considered variable is x and the affine expression is $x + 4$. The resulting polyhedron is the same square translated to the right. Moreover, if the affine transformation for the same variable x is $x + y$:

```
LinExpression coeff = x + y;
```

the resulting polyhedron is a parallelogram with the height equal to the side of the square and the oblique sides parallel to the line $x - y$. Instead, if we do not use an invertible transformation for the same variable; for example, the affine expression y:

```
LinExpression coeff = y;
```

the resulting polyhedron is a diagonal of the square.

Example 7

The following code shows the use of the function `affine_preimage`:

```
C_Polyhedron ph(2);
ph.add_constraint(x >= 0);
ph.add_constraint(x <= 3);
ph.add_constraint(y >= 0);
ph.add_constraint(y <= 3);
LinExpression coeff = x + 4;
ph.affine_preimage(x, coeff);
```

In this example the starting polyhedron, `var` and the affine expression and the denominator are the same as in Example 6, while the resulting polyhedron is again the same square, but translated to the left. Moreover, if the affine transformation for x is $x + y$

```
LinExpression coeff = x + y;
```

the resulting polyhedron is a parallelogram with the height equal to the side of the square and the oblique sides parallel to the line $x + y$. Instead, if we do not use an invertible transformation for the same variable x, for example, the affine expression y:

```
LinExpression coeff = y;
```

the resulting polyhedron is a line that corresponds to the y axis.

Example 8

For this example we use also the variables:

```
Variable z(2);
Variable w(3);
```

The following code shows the use of the function `remove_dimensions`:

```
GenSys gs;
gs.insert(point(3*x + y +0*z + 2*w));
C_Polyhedron ph(gs);
set<Variable> to_be_removed;
to_be_removed.insert(y);
to_be_removed.insert(z);
ph.remove_dimensions(to_be_removed);
```

The starting polyhedron is the singleton set $\left\{(3,1,0,2)^{\mathrm{T}}\right\} \subseteq \mathbb{R}^4$, while the resulting polyhedron is $\left\{(3,2)^{\mathrm{T}}\right\} \subseteq \mathbb{R}^2$. Be careful when removing dimensions *incrementally*: since dimensions are automatically renamed after each application of the `remove_dimensions` operator, unexpected results can be obtained. For instance, by using the following code we would obtain a different result:

```
set<Variable> to_be_removed1;
to_be_removed1.insert(y);
ph.remove_dimensions(to_be_removed1);
set<Variable> to_be_removed2;
to_be_removed2.insert(z);
ph.remove_dimensions(to_be_removed2);
```

In this case, the result is the polyhedron $\left\{(3,0)^{\mathrm{T}}\right\} \subseteq \mathbb{R}^2$: when removing the set of dimensions `to_be_removed2` we are actually removing variable w of the original polyhedron. For the same reason, the operator `remove_dimensions` is not idempotent: removing twice the same set of dimensions is never a no-op.

9.14.2 Member Enumeration Documentation

9.14.2.1 enum Parma_Polyhedra_Library::Polyhedron::Degenerate_Kind

Kinds of degenerate polyhedra.

Enumeration values:

UNIVERSE The universe polyhedron, i.e., the whole vector space.

EMPTY The empty polyhedron, i.e., the empty set.

9.14.3 Constructor & Destructor Documentation

9.14.3.1 Parma_Polyhedra_Library::Polyhedron::Polyhedron (Topology *topol*, dimension_type *num_dimensions*, Degenerate_Kind *kind*) `[protected]`

Builds a polyhedron having the specified properties.

Parameters:

topol The topology of the polyhedron;

num_dimensions The number of dimensions of the vector space enclosing the polyhedron;

kind Specifies whether the universe or the empty polyhedron has to be built.

9.14.3.2 Parma_Polyhedra_Library::Polyhedron::Polyhedron (Topology *topol*, const ConSys & *cs*) `[protected]`

Builds a polyhedron from a system of constraints.

The polyhedron inherits the space dimension of the constraint system.

Parameters:

topol The topology of the polyhedron;

cs The system of constraints defining the polyhedron.

Exceptions:

std::invalid_argument Thrown if the topology of `cs` is incompatible with `topol`.

9.14.3.3 Parma_Polyhedra_Library::Polyhedron::Polyhedron (Topology *topol*, ConSys & *cs*) [protected]

Builds a polyhedron recycling a system of constraints.

The polyhedron inherits the space dimension of the constraint system.

Parameters:

topol The topology of the polyhedron;

cs The system of constraints defining the polyhedron. It is not declared `const` because its data-structures will be recycled to build the polyhedron.

Exceptions:

std::invalid_argument Thrown if the topology of `cs` is incompatible with `topol`.

9.14.3.4 Parma_Polyhedra_Library::Polyhedron::Polyhedron (Topology *topol*, const GenSys & *gs*) [protected]

Builds a polyhedron from a system of generators.

The polyhedron inherits the space dimension of the generator system.

Parameters:

topol The topology of the polyhedron;

gs The system of generators defining the polyhedron.

Exceptions:

std::invalid_argument Thrown if if the topology of `gs` is incompatible with `topol`, or if the system of generators is not empty but has no points.

9.14.3.5 Parma_Polyhedra_Library::Polyhedron::Polyhedron (Topology *topol*, GenSys & *gs*) [protected]

Builds a polyhedron recycling a system of generators.

The polyhedron inherits the space dimension of the generator system.

Parameters:

topol The topology of the polyhedron;

gs The system of generators defining the polyhedron. It is not declared `const` because its data-structures will be recycled to build the polyhedron.

Exceptions:

std::invalid_argument Thrown if if the topology of `gs` is incompatible with `topol`, or if the system of generators is not empty but has no points.

9.14.3.6 template<typename Box> Parma_Polyhedra_Library::Polyhedron::Polyhedron (Topology *topol*, const Box & *box*) [protected]

Builds a polyhedron out of a generic, interval-based bounding box.

Parameters:

topol The topology of the polyhedron;

box The bounding box representing the polyhedron to be built.

Exceptions:

std::invalid_argument Thrown if `box` has intervals that are incompatible with `topol`.

The template class Box must provide the following methods.

```
dimension_type space_dimension() const
```

returns the dimension of the vector space enclosing the polyhedron represented by the bounding box.

```
bool is_empty() const
```

returns `true` if and only if the bounding box describes the empty set. The **is_empty()**(p. 105) method will always be called before the methods below. However, if **is_empty()**(p. 105) returns `true`, none of the functions below will be called.

```
bool get_lower_bound(dimension_type k, bool closed,
                     Integer& n, Integer& d) const
```

Let I the interval corresponding to the `k`-th dimension. If I is not bounded from below, simply return `false`. Otherwise, set `closed`, `n` and `d` as follows: `closed` is set to `true` if the the lower boundary of I is closed and is set to `false` otherwise; `n` and `d` are assigned the integers n and d such that the canonical fraction n/d corresponds to the greatest lower bound of I. The fraction n/d is in canonical form if and only if n and d have no common factors and d is positive, $0/1$ being the unique representation for zero.

```
bool get_upper_bound(dimension_type k, bool closed,
                     Integer& n, Integer& d) const
```

Let I the interval corresponding to the `k`-th dimension. If I is not bounded from above, simply return `false`. Otherwise, set `closed`, `n` and `d` as follows: `closed` is set to `true` if the the upper boundary of I is closed and is set to `false` otherwise; `n` and `d` are assigned the integers n and d such that the canonical fraction n/d corresponds to the least upper bound of I.

9.14.4 Member Function Documentation

9.14.4.1 Poly_Con_Relation Parma_Polyhedra_Library::Polyhedron::relation_with (const Constraint & *c*) const

Returns the relations holding between the polyhedron `*this` and the constraint `c`.

Exceptions:

std::invalid_argument Thrown if `*this` and constraint `c` are dimension-incompatible.

9.14.4.2 Poly_Gen_Relation Parma_Polyhedra_Library::Polyhedron::relation_with (const Generator & *g*) const

Returns the relations holding between the polyhedron `*this` and the generator `g`.

Exceptions:

std::invalid_argument Thrown if `*this` and generator `g` are dimension-incompatible.

9.14.4.3 bool Parma_Polyhedra_Library::Polyhedron::is_disjoint_from (const Polyhedron & *y*) const

Returns `true` if and only if `*this` and `y` are disjoint.

Exceptions:

std::invalid_argument Thrown if `x` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.4 bool Parma_Polyhedra_Library::Polyhedron::bounds_from_above (const LinExpression & *expr*) const

Returns `true` if and only if `expr` is bounded from above in `*this`.

Exceptions:

std::invalid_argument Thrown if `expr` and `*this` are dimension-incompatible.

9.14.4.5 bool Parma_Polyhedra_Library::Polyhedron::bounds_from_below (const LinExpression & *expr*) const

Returns `true` if and only if `expr` is bounded from below in `*this`.

Exceptions:

std::invalid_argument Thrown if `expr` and `*this` are dimension-incompatible.

9.14.4.6 bool Parma_Polyhedra_Library::Polyhedron::maximize (const LinExpression & *expr*, Integer & *sup_n*, Integer & *sup_d*, bool & *maximum*) const

Returns `true` if and only if `*this` is not empty and `expr` is bounded from above in `*this`, in which case the supremum value is computed.

Parameters:

expr The linear expression to be maximized subject to `*this`;

sup_n The numerator of the supremum value;

sup_d The denominator of the supremum value;

maximum `true` if and only if the supremum is also the maximum value.

Exceptions:

std::invalid_argument Thrown if `expr` and `*this` are dimension-incompatible.

If `*this` is empty or `expr` is not bounded from above, `false` is returned and `sup_n`, `sup_d` and `maximum` are left untouched.

9.14.4.7 bool Parma_Polyhedra_Library::Polyhedron::maximize (const LinExpression & *expr*, Integer & *sup_n*, Integer & *sup_d*, bool & *maximum*, const Generator **const *pppoint*) const

Returns `true` if and only if `*this` is not empty and `expr` is bounded from above in `*this`, in which case the supremum value and a point where `expr` reaches it are computed.

Parameters:

expr The linear expression to be maximized subject to `*this`;

sup_n The numerator of the supremum value;

sup_d The denominator of the supremum value;

maximum `true` if and only if the supremum is also the maximum value;

pppoint When nonzero and maximization succeeds, a pointer to a point or closure point where `expr` reaches its supremum value will be written at this address.

Exceptions:

std::invalid_argument Thrown if `expr` and `*this` are dimension-incompatible.

If `*this` is empty or `expr` is not bounded from above, `false` is returned and `sup_n`, `sup_d`, `maximum` and `pppoint` are left untouched.

9.14.4.8 bool Parma_Polyhedra_Library::Polyhedron::minimize (const LinExpression & *expr*, Integer & *inf_n*, Integer & *inf_d*, bool & *minimum*) const

Returns `true` if and only if `*this` is not empty and `expr` is bounded from below in `*this`, in which case the infimum value is computed.

Parameters:

expr The linear expression to be minimized subject to `*this`;

inf_n The numerator of the infimum value;

inf_d The denominator of the infimum value;

minimum `true` if and only if the infimum is also the minimum value.

Exceptions:

std::invalid_argument Thrown if `expr` and `*this` are dimension-incompatible.

If `*this` is empty or `expr` is not bounded from below, `false` is returned and `inf_n`, `inf_d` and `minimum` are left untouched.

9.14.4.9 bool Parma_Polyhedra_Library::Polyhedron::minimize (const LinExpression & *expr*, Integer & *inf_n*, Integer & *inf_d*, bool & *minimum*, const Generator **const *pppoint*) const

Returns `true` if and only if `*this` is not empty and `expr` is bounded from below in `*this`, in which case the infimum value and a point where `expr` reaches it are computed.

Parameters:

expr The linear expression to be minimized subject to `*this`;

inf_n The numerator of the infimum value;

inf_d The denominator of the infimum value;

minimum `true` if and only if the infimum is also the minimum value;

pppoint When nonzero and minimization succeeds, a pointer to a point or closure point where `expr` reaches its infimum value will be written at this address.

Exceptions:

std::invalid_argument Thrown if `expr` and `*this` are dimension-incompatible.

If `*this` is empty or `expr` is not bounded from below, `false` is returned and `inf_n`, `inf_d`, `minimum` and `pppoint` are left untouched.

9.14.4.10 bool Parma_Polyhedra_Library::Polyhedron::contains (const Polyhedron & *y*) const

Returns `true` if and only if `*this` contains `y`.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.11 bool Parma_Polyhedra_Library::Polyhedron::strictly_contains (const Polyhedron & *y*) const

Returns `true` if and only if `*this` strictly contains `y`.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.12 template<typename Box> void Parma_Polyhedra_Library::Polyhedron::shrink_-bounding_box (Box & *box*, Complexity_Class *complexity* = ANY) const

Uses `*this` to shrink a generic, interval-based bounding box.

Parameters:

box The bounding box to be shrunk;

complexity The complexity class of the algorithm to be used.

The template class Box must provide the following methods, whose return value, if any, is simply ignored.

```
set_empty()
```

causes the box to become empty, i.e., to represent the empty set.

```
raise_lower_bound(dimension_type k, bool closed,
                  const Integer& n, const Integer& d)
```

intersects the interval corresponding to the `k`-th dimension with $[n/d, +\infty)$ if `closed` is `true`, with $(n/d, +\infty)$ if `closed` is `false`.

```
lower_upper_bound(dimension_type k, bool closed,
                  const Integer& n, const Integer& d)
```

intersects the interval corresponding to the `k`-th dimension with $(-\infty, n/d]$ if `closed` is `true`, with $(-\infty, n/d)$ if `closed` is `false`.

The function `raise_lower_bound(k, closed, n, d)` will be called at most once for each possible value for `k` and for all such calls the fraction n/d will be in canonical form, that is, n and d have no common factors and d is positive, $0/1$ being the unique representation for zero. The same guarantee is offered for the function `lower_upper_bound(k, closed, n, d)`.

9.14.4.13 bool Parma_Polyhedra_Library::Polyhedron::OK (bool *check_not_empty* = `false`) const

Checks if all the invariants are satisfied.

Returns:

`true` if and only if `*this` satisfies all the invariants and either `check_not_empty` is `false` or `*this` is not empty.

Parameters:

check_not_empty `true` if and only if, in addition to checking the invariants, `*this` must be checked to be not empty.

The check is performed so as to intrude as little as possible. If the library has been compiled with run-time assertions enabled, error messages are written on `std::cerr` in case invariants are violated. This is useful for the purpose of debugging the library.

9.14.4.14 void Parma_Polyhedra_Library::Polyhedron::add_constraint (const Constraint & *c*)

Adds a copy of constraint `c` to the system of constraints of `*this` (without minimizing the result).

Exceptions:

std::invalid_argument Thrown if `*this` and constraint `c` are topology-incompatible or dimension-incompatible.

9.14.4.15 bool Parma_Polyhedra_Library::Polyhedron::add_constraint_and_minimize (const Constraint & *c*)

Adds a copy of constraint `c` to the system of constraints of `*this`, minimizing the result.

Returns:

`false` if and only if the result is empty.

Exceptions:

std::invalid_argument Thrown if `*this` and constraint `c` are topology-incompatible or dimension-incompatible.

9.14.4.16 void Parma_Polyhedra_Library::Polyhedron::add_generator (const Generator & *g*)

Adds a copy of generator `g` to the system of generators of `*this` (without minimizing the result).

Exceptions:

std::invalid_argument Thrown if `*this` and generator `g` are topology-incompatible or dimension-incompatible, or if `*this` is an empty polyhedron and `g` is not a point.

9.14.4.17 bool Parma_Polyhedra_Library::Polyhedron::add_generator_and_minimize (const Generator & *g*)

Adds a copy of generator `g` to the system of generators of `*this`, minimizing the result.

Returns:

false if and only if the result is empty.

Exceptions:

std::invalid_argument Thrown if *this and generator g are topology-incompatible or dimension-incompatible, or if *this is an empty polyhedron and g is not a point.

9.14.4.18 void Parma_Polyhedra_Library::Polyhedron::add_constraints (const ConSys & *cs*)

Adds a copy of the constraints in cs to the system of constraints of *this (without minimizing the result).

Parameters:

cs Contains the constraints that will be added to the system of constraints of *this.

Exceptions:

std::invalid_argument Thrown if *this and cs are topology-incompatible or dimension-incompatible.

9.14.4.19 void Parma_Polyhedra_Library::Polyhedron::add_recycled_constraints (ConSys & *cs*)

Adds the constraints in cs to the system of constraints of *this (without minimizing the result).

Parameters:

cs The constraint system that will be recycled, adding its constraints to the system of constraints of *this.

Exceptions:

std::invalid_argument Thrown if *this and cs are topology-incompatible or dimension-incompatible.

Warning:

The only assumption that can be made on cs upon successful or exceptional return is that it can be safely destroyed.

9.14.4.20 bool Parma_Polyhedra_Library::Polyhedron::add_constraints_and_minimize (const ConSys & *cs*)

Adds a copy of the constraints in cs to the system of constraints of *this, minimizing the result.

Returns:

false if and only if the result is empty.

Parameters:

cs Contains the constraints that will be added to the system of constraints of *this.

Exceptions:

std::invalid_argument Thrown if *this and cs are topology-incompatible or dimension-incompatible.

9.14.4.21 bool Parma_Polyhedra_Library::Polyhedron::add_recycled_constraints_and_minimize (ConSys & *cs*)

Adds the constraints in `cs` to the system of constraints of `*this`, minimizing the result.

Returns:
`false` if and only if the result is empty.

Parameters:
cs The constraint system that will be recycled, adding its constraints to the system of constraints of `*this`.

Exceptions:
std::invalid_argument Thrown if `*this` and `cs` are topology-incompatible or dimension-incompatible.

Warning:
The only assumption that can be made on `cs` upon successful or exceptional return is that it can be safely destroyed.

9.14.4.22 void Parma_Polyhedra_Library::Polyhedron::add_generators (const GenSys & *gs*)

Adds a copy of the generators in `gs` to the system of generators of `*this` (without minimizing the result).

Parameters:
gs Contains the generators that will be added to the system of generators of `*this`.

Exceptions:
std::invalid_argument Thrown if `*this` and `gs` are topology-incompatible or dimension-incompatible, or if `*this` is empty and the system of generators `gs` is not empty, but has no points.

9.14.4.23 void Parma_Polyhedra_Library::Polyhedron::add_recycled_generators (GenSys & *gs*)

Adds the generators in `gs` to the system of generators of `*this` (without minimizing the result).

Parameters:
gs The generator system that will be recycled, adding its generators to the system of generators of `*this`.

Exceptions:
std::invalid_argument Thrown if `*this` and `gs` are topology-incompatible or dimension-incompatible, or if `*this` is empty and the system of generators `gs` is not empty, but has no points.

Warning:
The only assumption that can be made on `gs` upon successful or exceptional return is that it can be safely destroyed.

9.14.4.24 bool Parma_Polyhedra_Library::Polyhedron::add_generators_and_minimize (const GenSys & *gs*)

Adds a copy of the generators in `gs` to the system of generators of `*this`, minimizing the result.

Returns:
`false` if and only if the result is empty.

Parameters:
gs Contains the generators that will be added to the system of generators of `*this`.

Exceptions:
std::invalid_argument Thrown if `*this` and `gs` are topology-incompatible or dimension-incompatible, or if `*this` is empty and the the system of generators `gs` is not empty, but has no points.

9.14.4.25 bool Parma_Polyhedra_Library::Polyhedron::add_recycled_generators_and_minimize (GenSys & *gs*)

Adds the generators in `gs` to the system of generators of `*this`, minimizing the result.

Returns:
`false` if and only if the result is empty.

Parameters:
gs The generator system that will be recycled, adding its generators to the system of generators of `*this`.

Exceptions:
std::invalid_argument Thrown if `*this` and `gs` are topology-incompatible or dimension-incompatible, or if `*this` is empty and the the system of generators `gs` is not empty, but has no points.

Warning:
The only assumption that can be made on `gs` upon successful or exceptional return is that it can be safely destroyed.

9.14.4.26 void Parma_Polyhedra_Library::Polyhedron::intersection_assign (const Polyhedron & *y*)

Assigns to `*this` the intersection of `*this` and `y`. The result is not guaranteed to be minimized.

Exceptions:
std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.27 bool Parma_Polyhedra_Library::Polyhedron::intersection_assign_and_minimize (const Polyhedron & *y*)

Assigns to `*this` the intersection of `*this` and `y`, minimizing the result.

Returns:

`false` if and only if the result is empty.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.28 void Parma_Polyhedra_Library::Polyhedron::poly_hull_assign (const Polyhedron & *y*)

Assigns to `*this` the poly-hull of `*this` and `y`. The result is not guaranteed to be minimized.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.29 bool Parma_Polyhedra_Library::Polyhedron::poly_hull_assign_and_minimize (const Polyhedron & *y*)

Assigns to `*this` the poly-hull of `*this` and `y`, minimizing the result.

Returns:

`false` if and only if the result is empty.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.30 void Parma_Polyhedra_Library::Polyhedron::poly_difference_assign (const Polyhedron & *y*)

Assigns to `*this` the **poly-difference**(p. 6) of `*this` and `y`. The result is not guaranteed to be minimized.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.31 void Parma_Polyhedra_Library::Polyhedron::affine_image (Variable *var*, const LinExpression & *expr*, const Integer & *denominator* = `Integer_one()`)

Assigns to `*this` the **affine image**(p. 8) of `*this` under the function mapping variable `var` to the affine expression specified by `expr` and `denominator`.

Parameters:

var The variable to which the affine expression is assigned;

expr The numerator of the affine expression;

denominator The denominator of the affine expression (optional argument with default value 1.)

Exceptions:

std::invalid_argument Thrown if `denominator` is zero or if `expr` and `*this` are dimension-incompatible or if `var` is not a dimension of `this`.

9.14.4.32 void Parma_Polyhedra_Library::Polyhedron::affine_preimage (Variable *var*, const LinExpression & *expr*, const Integer & *denominator* = `Integer_one()`)

Assigns to `*this` the **affine preimage**(p. 8) of `*this` under the function mapping variable `var` to the affine expression specified by `expr` and `denominator`.

Parameters:

var The variable to which the affine expression is substituted;

expr The numerator of the affine expression;

denominator The denominator of the affine expression (optional argument with default value 1.)

Exceptions:

std::invalid_argument Thrown if `denominator` is zero or if `expr` and `*this` are dimension-incompatible or if `var` is not a dimension of `*this`.

9.14.4.33 void Parma_Polyhedra_Library::Polyhedron::generalized_affine_image (Variable *var*, const Relation_Symbol *relsym*, const LinExpression & *expr*, const Integer & *denominator* = `Integer_one()`)

Assigns to `*this` the image of `*this` with respect to the **generalized affine transfer function**(p. 9) $\mathrm{var}' \bowtie \frac{\mathrm{expr}}{\mathrm{denominator}}$, where $\bowtie$ is the relation symbol encoded by `relsym`.

Parameters:

var The left hand side variable of the generalized affine transfer function;

relsym The relation symbol;

expr The numerator of the right hand side affine expression;

denominator The denominator of the right hand side affine expression (optional argument with default value 1.)

Exceptions:

std::invalid_argument Thrown if `denominator` is zero or if `expr` and `*this` are dimension-incompatible or if `var` is not a dimension of `*this` or if `*this` is a **C_Polyhedron**(p. 71) and `relsym` is a strict relation symbol.

9.14.4.34 void Parma_Polyhedra_Library::Polyhedron::generalized_affine_image (const LinExpression & *lhs*, const Relation_Symbol *relsym*, const LinExpression & *rhs*)

Assigns to `*this` the image of `*this` with respect to the **generalized affine transfer function**(p. 9) $\mathrm{lhs}' \bowtie \mathrm{rhs}$, where $\bowtie$ is the relation symbol encoded by `relsym`.

Parameters:

lhs The left hand side affine expression;

relsym The relation symbol;

rhs The right hand side affine expression.

Exceptions:

std::invalid_argument Thrown if `*this` is dimension-incompatible with `lhs` or `rhs` or if `*this` is a **C_Polyhedron**(p. 71) and `relsym` is a strict relation symbol.

9.14.4.35 void Parma_Polyhedra_Library::Polyhedron::time_elapse_assign (const Polyhedron & *y*)

Assigns to `*this` the result of computing the **time-elapse**(p. 9) between `*this` and `y`.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.36 void Parma_Polyhedra_Library::Polyhedron::BHRZ03_widening_assign (const Polyhedron & *y*, unsigned ∗ *tp* = 0)

Assigns to `*this` the result of computing the **BHRZ03-widening**(p. 10) between `*this` and `y`.

Parameters:

y A polyhedron that *must* be contained in `*this`;

tp An optional pointer to an unsigned variable storing the number of available tokens (to be used when applying the **widening with tokens**(p. 11) delay technique).

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.37 void Parma_Polyhedra_Library::Polyhedron::limited_BHRZ03_extrapolation_assign (const Polyhedron & *y*, const ConSys & *cs*, unsigned ∗ *tp* = 0)

Improves the result of the **BHRZ03-widening**(p. 10) computation by also enforcing those constraints in `cs` that are satisfied by all the points of `*this`.

Parameters:

y A polyhedron that *must* be contained in `*this`;

cs The system of constraints used to improve the widened polyhedron;

tp An optional pointer to an unsigned variable storing the number of available tokens (to be used when applying the **widening with tokens**(p. 11) delay technique).

Exceptions:

std::invalid_argument Thrown if `*this`, `y` and `cs` are topology-incompatible or dimension-incompatible.

9.14.4.38 void Parma_Polyhedra_Library::Polyhedron::bounded_BHRZ03_extrapolation_assign (const Polyhedron & *y*, const ConSys & *cs*, unsigned ∗ *tp* = 0)

Improves the result of the **BHRZ03-widening**(p. 10) computation by also enforcing those constraints in `cs` that are satisfied by all the points of `*this`, plus all the constraints of the form $\pm x \leq r$ and $\pm x < r$, with $r \in \mathbb{Q}$, that are satisfied by all the points of `*this`.

Parameters:

y A polyhedron that *must* be contained in `*this`;

cs The system of constraints used to improve the widened polyhedron;

tp An optional pointer to an unsigned variable storing the number of available tokens (to be used when applying the **widening with tokens**(p. 11) delay technique).

Exceptions:

std::invalid_argument Thrown if `*this`, `y` and `cs` are topology-incompatible or dimension-incompatible.

9.14.4.39 void Parma_Polyhedra_Library::Polyhedron::H79_widening_assign (const Polyhedron & *y*, unsigned ∗ *tp* = 0)

Assigns to `*this` the result of computing the **H79-widening**(p. 10) between `*this` and `y`.

Parameters:

y A polyhedron that *must* be contained in `*this`;

tp An optional pointer to an unsigned variable storing the number of available tokens (to be used when applying the **widening with tokens**(p. 11) delay technique).

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible or dimension-incompatible.

9.14.4.40 void Parma_Polyhedra_Library::Polyhedron::limited_H79_extrapolation_assign (const Polyhedron & *y*, const ConSys & *cs*, unsigned ∗ *tp* = 0)

Improves the result of the **H79-widening**(p. 10) computation by also enforcing those constraints in `cs` that are satisfied by all the points of `*this`.

Parameters:

y A polyhedron that *must* be contained in `*this`;

cs The system of constraints used to improve the widened polyhedron;

tp An optional pointer to an unsigned variable storing the number of available tokens (to be used when applying the **widening with tokens**(p. 11) delay technique).

Exceptions:

std::invalid_argument Thrown if `*this`, `y` and `cs` are topology-incompatible or dimension-incompatible.

9.14.4.41 void Parma_Polyhedra_Library::Polyhedron::bounded_H79_extrapolation_assign (const Polyhedron & *y*, const ConSys & *cs*, unsigned ∗ *tp* = 0)

Improves the result of the **H79-widening**(p. 10) computation by also enforcing those constraints in `cs` that are satisfied by all the points of `*this`, plus all the constraints of the form $\pm x \leq r$ and $\pm x < r$, with $r \in \mathbb{Q}$, that are satisfied by all the points of `*this`.

Parameters:

y A polyhedron that *must* be contained in `*this`;

cs The system of constraints used to improve the widened polyhedron;

tp An optional pointer to an unsigned variable storing the number of available tokens (to be used when applying the **widening with tokens**(p. 11) delay technique).

Exceptions:

std::invalid_argument Thrown if `*this`, `y` and `cs` are topology-incompatible or dimension-incompatible.

9.14.4.42 void Parma_Polyhedra_Library::Polyhedron::add_dimensions_and_embed (dimension_type *m*)

Adds `m` new dimensions and embeds the old polyhedron in the new space.

Parameters:

m The number of dimensions to add.

The new dimensions will be those having the highest indexes in the new polyhedron, which is characterized by a system of constraints in which the variables running through the new dimensions are not constrained. For instance, when starting from the polyhedron $\mathcal{P} \subseteq \mathbb{R}^2$ and adding a third dimension, the result will be the polyhedron

$$\left\{ (x, y, z)^{\mathrm{T}} \in \mathbb{R}^3 \;\middle|\; (x, y)^{\mathrm{T}} \in \mathcal{P} \right\}.$$

9.14.4.43 void Parma_Polyhedra_Library::Polyhedron::add_dimensions_and_project (dimension_type *m*)

Adds `m` new dimensions to the polyhedron and does not embed it in the new space.

Parameters:

m The number of dimensions to add.

The new dimensions will be those having the highest indexes in the new polyhedron, which is characterized by a system of constraints in which the variables running through the new dimensions are all constrained to be equal to 0. For instance, when starting from the polyhedron $\mathcal{P} \subseteq \mathbb{R}^2$ and adding a third dimension, the result will be the polyhedron

$$\left\{ (x, y, 0)^{\mathrm{T}} \in \mathbb{R}^3 \;\middle|\; (x, y)^{\mathrm{T}} \in \mathcal{P} \right\}.$$

9.14.4.44 void Parma_Polyhedra_Library::Polyhedron::concatenate_assign (const Polyhedron & *y*)

Assigns to `*this` the **concatenation**(p. 6) of `*this` and `y`, taken in this order.

Exceptions:

std::invalid_argument Thrown if `*this` and `y` are topology-incompatible.

9.14.4.45 void Parma_Polyhedra_Library::Polyhedron::remove_dimensions (const Variables_Set & *to_be_removed*)

Removes all the specified dimensions.

Parameters:

to_be_removed The set of **Variable**(p. 132) objects corresponding to the dimensions to be removed.

Exceptions:

std::invalid_argument Thrown if `*this` is dimension-incompatible with one of the **Variable**(p. 132) objects contained in `to_be_removed`.

9.14.4.46 void Parma_Polyhedra_Library::Polyhedron::remove_higher_dimensions (dimension_type *new_dimension*)

Removes the higher dimensions so that the resulting space will have dimension `new_dimension`.

Exceptions:

std::invalid_argument Thrown if `new_dimensions` is greater than the space dimension of `*this`.

9.14.4.47 template<typename PartialFunction> void Parma_Polyhedra_Library::Polyhedron::map_dimensions (const PartialFunction & *pfunc*)

Remaps the dimensions of the vector space according to a **partial function**(p. 7).

Parameters:

pfunc The partial function specifying the destiny of each dimension.

The template class PartialFunction must provide the following methods.

```
bool has_empty_codomain() const
```

returns `true` if and only if the represented partial function has an empty codomain (i.e., it is always undefined). The `has_empty_codomain()` method will always be called before the methods below. However, if `has_empty_codomain()` returns `true`, none of the functions below will be called.

```
dimension_type max_in_codomain() const
```

returns the maximum value that belongs to the codomain of the partial function. The `max_in_codomain()` method is called at most once.

```
bool maps(dimension_type i, dimension_type& j) const
```

Let f be the represented function and k be the value of `i`. If f is defined in k, then $f(k)$ is assigned to `j` and `true` is returned. If f is undefined in k, then `false` is returned. This method is called at most n times, where n is the dimension of the vector space enclosing the polyhedron.

The result is undefined if `pfunc` does not encode a partial function with the properties described in the **specification of the mapping operator**(p. 7).

9.14.4.48 void Parma_Polyhedra_Library::Polyhedron::expand_dimension (Variable *var*, dimension_type *m*)

Creates `m` copies of the dimension corresponding to `var`.

Parameters:

var The variable corresponding to the dimension to be replicated;

m The number of replica to be created.

Exceptions:

std::invalid_argument Thrown if `var` does not correspond to a dimension of the polyhedron.

If `*this` is n-dimensional, with $n > 0$, and $i < n$ is `var.id()`, then the i-th dimension is **expanded**(p. 7) to `m` new dimensions $n, n+1, \ldots, n+m-1$.

9.14.4.49 void Parma_Polyhedra_Library::Polyhedron::fold_dimensions (const Variables_Set & *to_be_folded*, Variable *var*)

Folds the dimensions in `to_be_folded` into `var`.

Parameters:

to_be_folded The set of **Variable**(p. 132) objects corresponding to the dimensions to be folded;

var The variable corresponding to the dimension that is the destination of the folding operation.

Exceptions:

std::invalid_argument Thrown if `*this` is dimension-incompatible with `var` or with one of the **Variable**(p. 132) objects contained in `to_be_folded`. Also thrown if `var` is contained in `to_be_folded`.

If `*this` is n-dimensional, with $n > 0$, $i < n$ is `var.id()`, `to_be_folded` is a set of variables whose `id()` is also less than n, and `var` is not a member of `to_be_folded`, then the dimensions corresponding to variables in `to_be_folded` are **folded**(p. 8) into dimension i.

9.14.4.50 void Parma_Polyhedra_Library::Polyhedron::swap (Polyhedron & *y*)

Swaps `*this` with polyhedron `y`. (`*this` and `y` can be dimension-incompatible.).

Exceptions:

std::invalid_argument Thrown if `x` and `y` are topology-incompatible.

9.14.5 Friends And Related Function Documentation

9.14.5.1 std::ostream & operator<< (std::ostream & *s*, const Polyhedron & *ph*) `[related]`

Output operator.

Writes a textual representation of `ph` on `s`: `false` is written if `ph` is an empty polyhedron; `true` is written if `ph` is a universe polyhedron; a minimized system of constraints defining `ph` is written otherwise, all constraints in one row separated by ", ".

9.14.5.2 bool operator== (const Polyhedron & *x*, const Polyhedron & *y*) `[related]`

Returns `true` if and only if `x` and `y` are the same polyhedron.

Note that `x` and `y` may be topology- and/or dimension-incompatible polyhedra: in those cases, the value `false` is returned.

9.14.5.3 bool operator!= (const Polyhedron & *x*, const Polyhedron & *y*) `[related]`

Returns `true` if and only if `x` and `y` are different polyhedra.

Note that `x` and `y` may be topology- and/or dimension-incompatible polyhedra: in those cases, the value `true` is returned.

9.15 Parma_Polyhedra_Library::PowerSet< CS > Class Template Reference

The powerset construction on constraint systems.

Public Member Functions

- **PowerSet** ()

  Default constructor.

- **PowerSet** (const **PowerSet** &y)

  Ordinary copy-constructor.

- **PowerSet** & **operator=** (const **PowerSet** &y)

  The assignment operator.

- void **swap** (**PowerSet** &y)

  Swaps `*this` *with* `y`.

- void **add_disjunct** (const CS &d)

  Adds to `*this` *the disjunct* `d`.

- void **upper_bound_assign** (const **PowerSet** &y)

  Assigns to `*this` *an upper bound of* `*this` *and* `y`.

- void **meet_assign** (const **PowerSet** &y)

  Assigns to `*this` *the meet of* `*this` *and* `y`.

- bool **definitely_entails** (const **PowerSet** &y) const

  Returns `true` *if* `*this` *definitely entails* `y`. *Returns* `false` *if* `*this` *may not entail* `y` *(i.e., if* `*this` *does not entail* `y` *or if entailment could not be decided).*

- bool **is_top** () const

  Returns `true` *if and only if* `*this` *is the top element of the powerset constraint system (i.e., it represents the universe).*

- bool **is_bottom** () const

  Returns `true` *if and only if* `*this` *is the bottom element of the powerset constraint system (i.e., it represents the empty set).*

- bool **OK** (bool disallow_bottom=false) const

  Checks if all the invariants are satisfied.

- void **collapse** ()

  If `*this` *is not empty (i.e., it is not the bottom element), it is reduced to a singleton obtained by computing an upper-bound of all the disjuncts.*

Protected Types

- typedef std::list< CS > **Sequence**

  A powerset is implemented as a sequence of elements.

Protected Member Functions

- void **omega_reduce** () const

  Erase from `*this` *all the non-maximal elements.*

- bool **is_omega_reduced** () const

  Returns `true` *if and only if* `*this` *does not contain non-maximal elements.*

Static Protected Member Functions

- void **add_non_bottom_disjunct** (**Sequence** &s, const CS &d, iterator &first, iterator last)

  Adds to `*this` *the disjunct* `d`*, assuming* `d` *is not the bottom element and ensuring partial omega-reduction.*

- void **add_non_bottom_disjunct** (**Sequence** &s, const CS &d)

  Adds to `*this` *the disjunct* `d`*, assuming* `d` *is not the bottom element.*

Protected Attributes

- **Sequence sequence**

  The sequence container holding powerset's elements.

- bool **reduced**

  If `true`*,* `*this` *is omega-reduced.*

Related Functions

(Note that these are not member functions.)

- bool **operator==** (const **PowerSet**< CS > &x, const **PowerSet**< CS > &y)

  Returns `true` *if and only if* `x` *and* `y` *are equivalent.*

- bool **operator!=** (const **PowerSet**< CS > &x, const **PowerSet**< CS > &y)

  Returns `true` *if and only if* `x` *and* `y` *are not equivalent.*

- std::ostream & **operator<<** (std::ostream &s, const **PowerSet**< CS > &x)

  Output operator.

- void **swap** (**Parma_Polyhedra_Library::PowerSet**< CS > &x, **Parma_Polyhedra_Library::PowerSet**< CS > &y)

  Specializes `std::swap`*.*

9.15.1 Detailed Description

template<typename CS> class Parma_Polyhedra_Library::PowerSet< CS >

The powerset construction on constraint systems.

This class offers a generic implementation of *powerset constraint systems* as defined in **[Bag98]**(p. 15).

9.15.2 Member Typedef Documentation

9.15.2.1 template<typename CS> typedef std::list<CS> Parma_Polyhedra_Library::PowerSet< CS >::Sequence `[protected]`

A powerset is implemented as a sequence of elements.

The particular sequence employed must support efficient deletion in any position and efficient back insertion.

9.15.3 Member Function Documentation

9.15.3.1 template<typename CS> void Parma_Polyhedra_Library::PowerSet< CS >::add_non_-bottom_disjunct (Sequence & *s*, const CS & *d*, iterator & *first*, iterator *last*) `[static, protected]`

Adds to `*this` the disjunct `d`, assuming `d` is not the bottom element and ensuring partial omega-reduction.

If `d` is not the bottom element and is not redundant with respect to the elements in positions between `first` and `last`, adds to `*this` the disjunct `d`, erasing all the elements in the above mentioned positions that are made omega-redundant by the addition of `d`.

9.16 Parma_Polyhedra_Library::Variable Class Reference

A dimension of the space.

Public Types

- typedef void **output_function_type** (std::ostream &s, const **Variable** &v)

  Type of output functions.

Public Member Functions

- **Variable** (dimension_type i)

  Builds the variable corresponding to the Cartesian axis of index `i`.

- dimension_type **id** () const

  Returns the index of the Cartesian axis associated to the variable.

Static Public Member Functions

- void **set_output_function** (**output_function_type** ∗p)

  Sets the output function to be used for printing **Variable**(p. 132) *objects.*

- **output_function_type** ∗ **get_output_function** ()

  Returns the pointer to the current output function.

Related Functions

(Note that these are not member functions.)

- std::ostream & **operator**<< (std::ostream &s, const **Variable** &v)

  Output operator.

- bool **less** (**Variable** v, **Variable** w)

  Defines a total ordering on variables.

9.16.1 Detailed Description

A dimension of the space.

An object of the class **Variable**(p. 132) represents a dimension of the space, that is one of the Cartesian axes. Variables are used as base blocks in order to build more complex linear expressions. Each variable is identified by a non-negative integer, representing the index of the corresponding Cartesian axis (the first axis has index 0).

Note that the "meaning" of an object of the class **Variable**(p. 132) is completely specified by the integer index provided to its constructor: be careful not to be mislead by C++ language variable names. For instance, in the following example the linear expressions `e1` and `e2` are equivalent, since the two variables `x` and `z` denote the same Cartesian axis.

```
Variable x(0);
Variable y(1);
Variable z(0);
LinExpression e1 = x + y;
LinExpression e2 = y + z;
```

9.17 Parma_Polyhedra_Library::Variable::Compare Struct Reference

Binary predicate defining the total ordering on variables.

Public Member Functions

- bool **operator()** (**Variable** x, **Variable** y) const

  Returns `true` *if and only if* `x` *comes before* `y`.

9.17.1 Detailed Description

Binary predicate defining the total ordering on variables.

10 PPL Page Documentation

10.1 GNU General Public License

Version 2, June 1991

```
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